The Clique Number of Generalized Hamming Graphs

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Contents

Chapter 1

Introduction

Let $A_1, ..., A_n$ be non-empty, finite sets (alphabets). The elements of the Cartesian product

$$V = A_1 \times ... \times A_n = \{(x_1, ..., x_n) : \ x_i \in A_i \text{ for } i = 1, ..., n\}$$

are considered as words z of length n with entries $z(i) = x_i$ from the alphabets A_i, $i = 1, ..., n$. The *Hamming distance* $d(y, z)$ of words $y, z \in V$ is the number of positions, in which their entries differ,

$$d(y, z) = |\{i : \ y(i) \neq z(i), 1 \leq i \leq n\}|.$$

Let D be a non empty subset of $\{1, 2, ..., n\}$ and $|A_i| = m_i$ for $i = 1, ..., n$. The (generalized) *Hamming graph* $HG(A_1, ..., A_n; D)$ has vertex set V and edge set

$$E = \{\{x, y\} : \ x, y \in V, d(x, y) \in D\}.$$

We arrive at the usual definition of Hamming graphs, which we now call 'ordinary Hamming graphs', if $D = \{1\}$ (see [5]).
The structure of $HG(A_1, ..., A_n; D)$ does not depend on the special nature of the elements of the alphabets A_i, but only on their cardinalities $|A_i| = m_i$, $i = 1, ..., n$. Therefore, we write

$$HG(A_1, ..., A_n; D) = HG(m_1, ..., m_n; D)$$

unless otherwise stated, our standard alphabet A_i is

$$A_i = \mathbb{Z}_{m_i} = \{0, 1, ..., m_i - 1\}.$$

By the addition of integers modulo m_i the alphabet $\mathbb{Z}_{m_i}$ becomes a group and V may be considered as the direct product (direct sum) of the cyclic

groups $\mathbb{Z}_{m_i}$, $i = 1, ..., n$.
The *zero word* $\overline{0} \in V$ has all entries equal to zero. The *weight* $w(z)$ of $z \in V$ is the number of nonzero entries of z. We have

$$w(z) = d(\overline{0}, z),\ d(x, y) = w(x - y) \text{ for } x, y, z \in V.$$

Hamming graphs are Cayley graphs, [3]. The (undirected) *Cayley graph* $Cay(H, S)$ has as its vertex set the elements of the (here: additive) group H. The set S is a subset of H, which does not contain the zero element of H and has the property $S \cup (-S) = S$. Vertices $x, y \in H$ are connected by an edge, if $x - y \in S$.
The Hamming graph $HG(m_1, ..., m_n; D)$ has the additive group

$$V = \mathbb{Z}_{m_1} \otimes ... \otimes \mathbb{Z}_{m_n}$$

as its vertex set. To generate its edge set define

$$S = \{x \in V :\ w(x) \in D\}.$$

Then we have $\overline{0} \notin S$, $x \in S$ implies $-x \in S$ and

$$HG(m_1, ..., m_n; D) = Cay(V, S).$$

The group of the Cayley graph induces a group of automorphisms acting transitively on the vertices of the graph. Cayley graphs $Cay(H, S)$ are *vertex transitive*, i.e. for any two vertices $x, y \in H$ there is an automorphism φ with $y = \varphi(x)$. Vertices of vertex transitive graphs have isomorphic neighborhoods. Therefore, these graphs are regular. The Cayley graph $Cay(H, S)$ is regular of degree $|S|$.

A *clique* in a graph G is a complete subgraph of G, i.e. a subgraph, in which any two vertices are connected by an edge. The *clique number* $\omega(G)$ is the largest number of vertices in a clique of G.

The following proposition is a consequence of the vertex transitivity of $HG(m_1, ..., m_n; D)$.

Proposition 1.1. *The Hamming graph $HG(m_1, ..., m_n; D)$ has a maximal clique C, $\omega(HG(m_1, ..., m_n; D)) = |C|$, which contains the zero word $\overline{0}$.*

The standard Hamming graphs we are going to investigate here have the same alphabet $\mathbb{Z}_m$ in each position, i.e. $m_1 = ... = m_n = m$. Moreover, the set of distances D consists of all distances from 1 up to a positive integer $d \leq n$, $D = \{1, 2, ..., d\}$. We simplify our notation,

$$HG(\underbrace{m, ..., m}_{n \text{ positions}}; \{1, ..., d\}) = HG(m, n, d).$$

So $HG(m, n, d)$ has vertex set $V = {\mathbb{Z}_m}^n$. Distinct vertices x and y are conected by an edge, if their Hamming distance $d(x, y)$ is at most d.

The main subject of this thesis is the clique number $\omega(HG(m, n, d))$. Clearly, all words of weight at most $\lfloor \frac{d}{2} \rfloor$ induce a clique in $HG(m, n, d)$. If d is odd then this clique may be extended by all words of weight $\frac{d+1}{2}$ with a nonzero entry in the first position. So one can easily deduce the following lower bound $\omega_0(m, n, d)$ for $\omega(HG(m, n, d))$.

Proposition 1.2. *Let $d = 2t + \varepsilon \geq 1$, $\varepsilon \in \{0, 1\}$, $t \in \{0\} \cup \mathbb{N}$, $\mathbb{N} = \{1, 2, ...\}$, $m, n \in \mathbb{N}$, $n \geq d$, $m \geq 2$. Then we have*

$$\omega(HG(m, n, d)) \geq \omega_0(m, n, d) = \sum_{j=0}^{t} \binom{n}{j} (m-1)^j + \varepsilon \binom{n-1}{t} (m-1)^{t+1}.$$

We believe that $\omega(HG(m, n, d))$ coincides with the lower bound in Proposition 1.2, if n is sufficient large.

ω_0-Conjecture:
For fixed parameters $m, d \in \mathbb{N}$ there is $n_0 = n_0(m, d) \in \mathbb{N}$ such that

$$\omega(HG(m, n, d)) = \omega_0(m, n, d) \quad \text{for every } n \geq n_0.$$

Our main results state that the ω_0-conjecture is true for $m = 2$, i.e. for binary Hamming graphs, and for $d \leq 6$.

Theorem 1.1.

1. *For positive integers m, d, $m \geq 2$, $d \leq 6$, there is $n_0 \in \mathbb{N}$ (depending only on d and m) such that for every $n \geq n_0$*

$$\omega(HG(m, n, d)) = \omega_0(m, n, d).$$

2. *For every positive integer d there is $n_0 \in \mathbb{N}$ (depending only on d) such that for every $n \geq n_0$*

$$\omega(HG(2, n, d)) = \omega_0(2, n, d).$$

The proofs frequently use binomial identities, which are studied in chapter 2. The expression for $\omega_0(m,n,d)$ indicates that there are principal differences between Hamming graphs with even or odd distance parameter d. So we devote different chapters, 3 and 4, to the even and odd case. In these chapters the ω_0-conjecture is proved up to $d=6$ and the main tools are developed to prove the ω_0-conjecture for binary Hamming graphs in chapter 5. Finally, in chapter 6 we state some further results on the clique number and also on the chromatic number of $HG(m,n,d)$ and of its complementary graph $\overline{HG}(m,n,d)$.

The generalized Hamming graph $HG(m,n,d)$ is defined in [7] as the d-th distance power of the ordinary Hamming graph $H(m,n) = {K_m}^n$, which is the n-fold Cartesian product [5] of the complete graph K_m. The *distance power* $G^{(d)}$ of an undirected graph G is obtained from G by drawing an edge between any two distinct vertices x, y of G at distance $d(x,y) \leq d$. So $H^{(d)}(m,n)$ in [7] is another notation for $HG(m,n,d)$.

In the literature little effort has been made to determine the clique number of $HG(m,n,d)$. More attention has been payed to the chromatic number $\chi(HG(m,n,d))$ [6-11,14]. Here $\omega_0(m,n,d)$ appears as a lower bound for the chromatic number. A proper coloring of $HG(m,n,d)$ is sometimes called a *distance d coloring of ${K_m}^n$*. Especially, distance d colorings of the hypercube ${K_2}^n$ or equivalently $\chi(HG(2,n,d))$ have been investigated. But only very few exact values of $\chi(HG(m,n,d))$ are known [9,10].

Definitions from graph theory omitted in this thesis can be found in [13].

Chapter 2

Identities for Binomial Coefficients

In this chapter we will prove lemmas for binomial coefficients that will be used in later chapters.

Let n and k be nonnegative integers, with $0 \leq k \leq n$. The *binomial coefficient* is defined to be the number of k-element subsets of a set of n elements. This number is denoted by $\binom{n}{k}$ and

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k(k-1)...1} = \frac{n!}{k!(n-k)!}$$

is the formula for the binomial coefficient. The definition of the binomial coefficient $\binom{n}{k}$ actually makes sense for any non-negative integers n and k: if $k > n$, then there are no k-subsets of an n-set, and $\binom{n}{k} = 0$. Also if $k < 0$, we define $\binom{n}{k} = 0$. Note that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$ and $\binom{n}{1} = n$.

The following fundamental facts about binomial coefficient are well known, see e.g. [1].

Proposition 2.1.

1. $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$ *Binomial Theorem*
2. $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ *Addition Theorem*
3. $\binom{n}{k} = \binom{n}{n-k}$. *Symmetry*

We remark that the Binomial Theorem is true in every commutative ring with a unit element.

The following lemma consists of some direct consequences of the fundamental properties of binomial coefficients.

Lemma 2.1. *For nonnegative integers $a, b, k, n,\ n \geq 1$, and an arbitrary integer c the following equations hold.*

1. $$\sum_{j=0}^{a} \binom{a}{j}\binom{b}{c-j} = \sum_{j=0}^{c} \binom{a}{j}\binom{b}{c-j} = \binom{a+b}{c},$$
2. $$2\sum_{j=0}^{k} \binom{n}{2j} = \sum_{j=0}^{2k} \binom{n}{j} + \binom{n-1}{2k},$$
3. $$2\sum_{j=0}^{k} \binom{n}{2j+1} = \sum_{j=0}^{2k+1} \binom{n}{j} + \binom{n-1}{2k+1},$$
4. $$2\sum_{k=0}^{b} \binom{n}{a-2k} = \sum_{j=a-2b-1}^{a} \binom{n}{j} + \binom{n-1}{a} - \binom{n-1}{a-2b-2},$$
5. $$\sum_{j=0}^{t} \binom{2t+2}{j} = 2^{2t+1} - \binom{2t+1}{t}.$$

Proof.

1. The equality of the two sums is easily established. So it suffices to prove

$$\sum_{j=0}^{a} \binom{a}{j}\binom{b}{c-j} = \binom{a+b}{c}.$$

If $c < 0$, then $\binom{a+b}{c} = 0$. Also $\binom{b}{c-j} = 0$ for every $j \geq 0$ and both sides of the equation have value equal to zero. So we may assume $c \geq 0$.
If $c > a+b$, then $c - j > b$ for every j, $0 \leq j \leq a$, and again both sides of the equation have value equal to zero. So we may assume $0 \leq c \leq a+b$.
We apply the binomial theorem to the identity

$$(1+x)^a(1+x)^b = (1+x)^{a+b}.$$

Thus we get

$$\sum_{j=0}^{a} \binom{a}{j} x^j \sum_{j'=0}^{b} \binom{b}{j'} x^{j'} = \sum_{k=0}^{a+b} \left(\sum_{j=0}^{a} \binom{a}{j}\binom{b}{k-j} \right) x^k = \sum_{k=0}^{a+b} \binom{a+b}{k} x^k.$$

Comparing coefficients of x^c confirms the assertion.

2. We use induction on k. For the basis step $k = 0$, we have on the left

$$2\sum_{j=0}^{0}\binom{n}{2j} = 2\binom{n}{0} = 2,$$

and on the right

$$\sum_{j=0}^{0}\binom{n}{j} + \binom{n-1}{0} = \binom{n}{0} + \binom{n-1}{0} = 2.$$

For the inductive step, suppose that the result is true for some $k \geq 0$. We consider

$$2\sum_{j=0}^{k+1}\binom{n}{2j} = 2\sum_{j=0}^{k}\binom{n}{2j} + 2\binom{n}{2(k+1)}.$$

By the induction hypothesis, this equation can be written as

$$\begin{aligned} 2\sum_{j=0}^{k+1}\binom{n}{2j} &= \sum_{j=0}^{2k}\binom{n}{j} + \binom{n-1}{2k} + 2\binom{n}{2k+2} \\ &= \sum_{j=0}^{2k}\binom{n}{j} + \binom{n-1}{2k} + \binom{n}{2k+2} + \binom{n}{2k+2}. \end{aligned}$$

By Proposition 2.1(2) we have

$$\binom{n}{2k+2} = \binom{n-1}{2k+1} + \binom{n-1}{2k+2}$$

and we conclude

$$2\sum_{j=0}^{k+1}\binom{n}{2j} = \sum_{j=0}^{2k}\binom{n}{j} + \left[\binom{n-1}{2k} + \binom{n-1}{2k+1}\right] + \binom{n-1}{2k+2} + \binom{n}{2k+2}.$$

Again we use Proposition 2.1(2) to get

$$\begin{aligned} 2\sum_{j=0}^{k+1}\binom{n}{2j} &= \sum_{j=0}^{2k}\binom{n}{j} + \binom{n}{2k+1} + \binom{n}{2k+2} + \binom{n-1}{2k+2} \\ &= \sum_{j=0}^{2k+2}\binom{n}{j} + \binom{n-1}{2k+2}. \end{aligned}$$

So the result is true for $k+1$ and by the principle of induction it is true for all integers $k \geq 0$.

3. We use induction on k. For the basis step $k=0$, we have on the left

$$2\sum_{j=0}^{0}\binom{n}{2j+1} = 2\binom{n}{1} = 2n,$$

and on the right

$$\sum_{j=0}^{1}\binom{n}{j} + \binom{n-1}{1} = \binom{n}{0} + \binom{n}{1} + \binom{n-1}{1} = 1+n+n-1 = 2n.$$

For the inductive step, suppose that the result is true for some $k \geq 0$. We consider

$$2\sum_{j=0}^{k+1}\binom{n}{2j+1} = 2\sum_{j=0}^{k}\binom{n}{2j+1} + 2\binom{n}{2(k+1)+1}.$$

By the induction hypothesis, this equation can be written as

$$\begin{aligned} 2\sum_{j=0}^{k+1}\binom{n}{2j+1} &= \sum_{j=0}^{2k+1}\binom{n}{j} + \binom{n-1}{2k+1} + 2\binom{n}{2k+3} \\ &= \sum_{j=0}^{2k+1}\binom{n}{j} + \binom{n-1}{2k+1} + \binom{n}{2k+3} + \binom{n}{2k+3}. \end{aligned}$$

By Proposition 2.1(2) we have

$$\binom{n}{2k+3} = \binom{n-1}{2k+2} + \binom{n-1}{2k+3}$$

and we conclude

$$2\sum_{j=0}^{k+1}\binom{n}{2j+1} = \sum_{j=0}^{2k+1}\binom{n}{j} + \left[\binom{n-1}{2k+1} + \binom{n-1}{2k+2}\right] + \binom{n-1}{2k+3} + \binom{n}{2k+3}.$$

Again we use Proposition 2.1(2) to get

$$\begin{aligned} 2\sum_{j=0}^{k+1}\binom{n}{2j+1} &= \sum_{j=0}^{2k+1}\binom{n}{j} + \binom{n}{2k+2} + \binom{n}{2k+3} + \binom{n-1}{2k+3} \\ &= \sum_{j=0}^{2k+3}\binom{n}{j} + \binom{n-1}{2k+3}. \end{aligned}$$

So the result is true for $k+1$ and by the principle of induction it is true for all integers $k \geq 0$.

4. We have

$$2\sum_{k=0}^{b}\binom{n}{a-2k} = 2\sum_{\substack{a-2b\leq h\leq a\\ h\equiv a \bmod 2}}\binom{n}{h}. \tag{2.1}$$

If $a-2b\leq 1$ then by part 2 and part 3, equation (2.1) can be written as

$$\begin{aligned}2\sum_{k=0}^{b}\binom{n}{a-2k} &= 2\sum_{\substack{0\leq h\leq a\\ h\equiv a \bmod 2}}\binom{n}{h}\\ &= \sum_{j=0}^{a}\binom{n}{j}+\binom{n-1}{a}.\end{aligned}$$

For $a-2b\leq 1$ this coincides with the right side of the equation in part 4.

So we may assume $a-2b\geq 2$. Now (2.1) can be written as

$$2\sum_{k=0}^{b}\binom{n}{a-2k} = 2\sum_{\substack{0\leq h\leq a\\ h\equiv a \bmod 2}}\binom{n}{h} - 2\sum_{\substack{0\leq h\leq a-2b-2\\ h\equiv a \bmod 2}}\binom{n}{h}.$$

The last sums can be evaluated by part 2 and part 3. Therefore

$$\begin{aligned}2\sum_{k=0}^{b}\binom{n}{a-2k} &= \sum_{j=0}^{a}\binom{n}{j}+\binom{n-1}{a}-\sum_{j=0}^{a-2b-2}\binom{n}{j}-\binom{n-1}{a-2b-2}\\ &= \sum_{j=a-2b-1}^{a}\binom{n}{j}+\binom{n-1}{a}-\binom{n-1}{a-2b-2}.\end{aligned}$$

5. We have by Proposition 2.1(1)

$$2^{2t+2} = \sum_{j=0}^{2t+2}\binom{2t+2}{j} = \sum_{j=0}^{t}\binom{2t+2}{j}+\sum_{j=t+2}^{2t+2}\binom{2t+2}{j}+\binom{2t+2}{t+1}. \tag{2.2}$$

The last two sums coincide by symmetry of binomial coefficients. Moreover, we have

$$\binom{2t+2}{t+1} = \binom{2t+1}{t}+\binom{2t+1}{t+1} = 2\binom{2t+1}{t}.$$

Inserting in (2.2) yields

$$2^{2t+2} = 2\sum_{j=0}^{t}\binom{2t+2}{j} + 2\binom{2t+1}{t},$$

$$\sum_{j=0}^{t}\binom{2t+2}{j} = 2^{2t+1} - \binom{2t+1}{t}.$$

□

Lemma 2.2. *For integers $a, n, r,\ 0 \le r \le n$, the following equation holds.*

$$\sum_{j=0}^{r-1}\sum_{h=r-j}^{r}\binom{r}{h}\binom{n-r}{a-2j-h} = \sum_{j=0}^{r-1}\sum_{h=0}^{r-1-j}\binom{r}{h}\binom{n-r}{a-r-2j-h}.$$

Proof. The equation is trivially true for $r = 0$. So we may assume $r \ge 1$. We consider

$$S = \sum_{j=0}^{r-1}\sum_{h=r-j}^{r}\binom{r}{h}\binom{n-r}{a-2j-h}.$$

Substituting $h = r - j + g$, $0 \le g \le j$, yields

$$S = \sum_{j=0}^{r-1}\sum_{g=0}^{j}\binom{r}{r-j+g}\binom{n-r}{a-2j-r+j-g}.$$

As $\binom{r}{r-j+g} = 0$ for $g > j$ and as $g \le j \le r-1$, we have

$$\begin{aligned} S &= \sum_{j=0}^{r-1}\sum_{g=0}^{r-1}\binom{r}{r-j+g}\binom{n-r}{a-r-j-g} \\ &= \sum_{g=0}^{r-1}\sum_{j=0}^{r-1}\binom{r}{r-j+g}\binom{n-r}{a-r-j-g} \\ &= \sum_{g=0}^{r-1}\sum_{j=g}^{r-1}\binom{r}{j-g}\binom{n-r}{a-r-2g-(j-g)}. \quad \text{(Proposition 2.1(3))} \end{aligned}$$

Substituting $h = j - g$, $0 \le h \le r - 1 - g$, we get

$$S = \sum_{g=0}^{r-1}\sum_{h=0}^{r-1-g}\binom{r}{h}\binom{n-r}{a-r-2g-h}.$$

Changing parameter g to j proves

$$S = \sum_{j=0}^{r-1} \sum_{h=0}^{r-1-j} \binom{r}{h} \binom{n-r}{a-r-2j-h}.$$

□

Lemma 2.3. *Let n, r, t be integers, $0 \leq r \leq n$. Define a_i, $i \in \mathbb{N}$, recursively by*

$$a_1 = \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r}{t-r-2k+j},$$

and

$$a_{i+1} = \sum_{j=0}^{r-1} \binom{n}{t-ir-2j} - a_i.$$

Then for $i \geq 1$ we have

$$a_i = \sum_{j=0}^{r-1} \sum_{h=r-j}^{r} \binom{r}{h} \binom{n-r}{t-(i-1)r-2j-h}.$$

Proof. As the assertion is trivially true for $r = 0$, we may assume $r \geq 1$. All relevant terms become 0 for $t \leq 0$. So we may also assume $t \geq 1$. We use induction on i.
(Induction basis) For $i = 1$ the result is true, because a_1 can be written as

$$\begin{aligned} a_1 &= \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{k-j} \binom{n-r}{t-r-2k+k-j} \\ &= \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{r-k+j} \binom{n-r}{t-2k-(r-k+j)}. \end{aligned}$$

Substituting $h = r-k+j$, $r-k \leq h \leq r$, yields

$$a_1 = \sum_{k=0}^{r-1} \sum_{h=r-k}^{r} \binom{r}{h} \binom{n-r}{t-2k-h}.$$

Changing parameter k to j confirms

$$a_1 = \sum_{j=0}^{r-1} \sum_{h=r-j}^{r} \binom{r}{h} \binom{n-r}{t-2j-h}.$$

(Induction hypothesis) Suppose that the result for a_i is true for some $i \geq 1$, that is,

$$a_i = \sum_{j=0}^{r-1} \sum_{h=r-j}^{r} \binom{r}{h} \binom{n-r}{t-(i-1)r-2j-h}.$$

By Lemma 2.2, a_i can be written

$$a_i = \sum_{j=0}^{r-1} \sum_{h=0}^{r-1-j} \binom{r}{h} \binom{n-r}{t-ir-2j-h}. \tag{2.3}$$

Then by recursive definition and Lemma 2.1(1) and by equation (2.3) we have

$$\begin{gathered} a_{i+1} = \sum_{j=0}^{r-1} \binom{n}{t-ir-2j} - a_i \\ = \\ \sum_{j=0}^{r-1} \sum_{h=0}^{r} \binom{r}{h} \binom{n-r}{t-ir-2j-h} - \sum_{j=0}^{r-1} \sum_{h=0}^{r-1-j} \binom{r}{h} \binom{n-r}{t-ir-2j-h} \\ = \\ \sum_{j=0}^{r-1} \left[\sum_{h=0}^{r} \binom{r}{h} \binom{n-r}{t-ir-2j-h} - \sum_{h=0}^{r-1-j} \binom{r}{h} \binom{n-r}{t-ir-2j-h} \right] \\ = \\ \sum_{j=0}^{r-1} \sum_{h=r-j}^{r} \binom{r}{h} \binom{n-r}{t-ir-2j-h}. \end{gathered}$$

So the result is true for a_{i+1} and by the principle of induction it is true for all integers $i \geq 1$. □

Lemma 2.4. *For integers n, r, t, $0 \leq r \leq n$, the following equation holds.*

$$\sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r+1}{t-r-2k+j} = \sum_{j=t-2r+1}^{t-r} \binom{n}{j}. \tag{2.4}$$

Proof. The Lemma is trivially true for $r = 0$. For $r = 1$ we have $\binom{n}{t-1}$ on both sides of equation (2.4). So we may assume $r \geq 2$. As equation (2.4) becomes trivial for $t \leq 0$, we may also assume $t \geq 1$.
Let S_1 be the left side of equation (2.4):

$$S_1 = \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r+1}{t-r-2k+j}.$$

Define recursively for $i \geq 1$:

$$S_{i+1} = \sum_{j=0}^{r-1} \binom{n+1}{t-ir-2j} - S_i.$$

We replace i by $i+1$,

$$S_{i+2} = \sum_{j=0}^{r-1} \binom{n+1}{t-(i+1)r-2j} - S_{i+1},$$

and solve the last two equations for S_i:

$$\begin{aligned} S_i &= \sum_{j=0}^{r-1} \left(\binom{n+1}{t-ir-2j} - \binom{n+1}{t-(i+1)r-2j} \right) + S_{i+2} \\ &= \sum_{j=0}^{r-1} \left(\binom{n}{t-ir-2j} + \binom{n}{t-ir-2j-1} \right) \quad \text{(Proposition 2.1(2))} \\ &\quad - \sum_{j=0}^{r-1} \left(\binom{n}{t-(i+1)r-2j} + \binom{n}{t-(i+1)r-2j-1} \right) + S_{i+2}, \end{aligned}$$

therefore

$$S_i = \sum_{h=0}^{2r-1} \binom{n}{t-ir-h} - \sum_{h=0}^{2r-1} \binom{n}{t-(i+1)r-h} + S_{i+2}. \tag{2.5}$$

Equation (2.5) implies for every odd integer $k \geq 1$:

$$S_1 = \sum_{\substack{1 \leq i \leq k \\ i \text{ odd}}} \sum_{h=0}^{2r-1} \binom{n}{t-ir-h} - \sum_{\substack{1 \leq i \leq k \\ i \text{ odd}}} \sum_{h=0}^{2r-1} \binom{n}{t-(i+1)r-h} + S_{k+2},$$

$$S_1 = \sum_{j=t-(k+2)r+1}^{t-r} \binom{n}{j} - \sum_{j=t-(k+3)r+1}^{t-2r} \binom{n}{j} + S_{k+2}. \tag{2.6}$$

Now we utilize Lemma 2.3 to get an explicit formula for S_{k+2}.

$$S_{k+2} = \sum_{j=0}^{r-1} \sum_{h=r-j}^{r} \binom{r}{h} \binom{n+1-r}{t-(k+1)r-2j-h}.$$

If we choose the odd integer k such that $(k+1)r \geq t+1$, then $S_{k+2} = 0$ and the sums in (2.6) may start with $j = 0$. So we receive

$$S_1 = \sum_{j=0}^{t-r} \binom{n}{j} - \sum_{j=0}^{t-2r} \binom{n}{j} = \sum_{j=t-2r+1}^{t-r} \binom{n}{j}.$$

□

Lemma 2.5. *Let n, r, t be integers, $1 \leq r \leq n-1$. Define a, b by*

$$\begin{aligned} a &= \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r}{t-r-2k+j}, \\ b &= \sum_{k=0}^{r-2} \sum_{j=0}^{k} \binom{r-1}{j} \binom{n-r+1}{t-r+1-2k+j}. \end{aligned}$$

Then the following equation holds.

$$a + b = \sum_{j=t-2r+1}^{t-r+1} \binom{n}{j} - \binom{n-1}{t-2r} - \binom{n}{t-2r+2}. \tag{2.7}$$

Proof. First let $r = 1$. In this case we have

$$a = \binom{n-1}{t-1}, \; b = 0.$$

The right side of (2.7) becomes

$$\binom{n}{t-1} + \binom{n}{t} - \binom{n-1}{t-2} - \binom{n}{t} = \binom{n-1}{t-1} + \binom{n-1}{t-2} - \binom{n-1}{t-2} = \binom{n-1}{t-1},$$

which proves (2.7) for $r = 1$. So we may assume $r \geq 2$. As for $t \leq 0$ all terms in (2.7) become zero, we may also assume $t \geq 1$.

We evaluate a by Lemma 2.4 inserting $n-1$ for n. Furthermore, we observe that b results from a by replacing r by $r-1$. Thus

$$\begin{aligned} a + b &= \sum_{j=t-2r+1}^{t-r} \binom{n-1}{j} + \sum_{j=t-2r+3}^{t-r+1} \binom{n-1}{j} \\ &= \sum_{j=t-2r+1}^{t-r} \binom{n-1}{j} + \sum_{j=t-2r+4}^{t-r+2} \binom{n-1}{j-1} \\ &= \sum_{j=t-2r+1}^{t-r+1} \left(\binom{n-1}{j} + \binom{n-1}{j-1} \right) - \binom{n-1}{t-r+1} + \binom{n-1}{t-r+1} \\ &\quad - \binom{n-1}{t-2r} - \binom{n-1}{t-2r+1} - \binom{n-1}{t-2r+2}, \end{aligned}$$

and we have by Proposition 2.1(2)

$$a+b=\sum_{j=t-2r+1}^{t-r+1}\binom{n}{j}-\binom{n-1}{t-2r}-\binom{n}{t-2r+2}.$$

$\square$

Chapter 3

Hamming Graphs with Even Distance

This chapter is dedicated to Hamming graphs $HG(m, n, d)$ with even distance parameter d and their clique numbers. We show that the ω_0-conjecture is true for Hamming graphs which have even distance at most 6. Some of the theorems and lemmas extend to the odd distance case and will also be used in later chapters.

First we will introduce some notations. For $x \in {\mathbb{Z}_m}^n$ we define the *support*

$$supp(x) = \{i : x(i) \neq 0 \ \text{ for } \ 1 \leq i \leq n\}.$$

Let $x, y \in {\mathbb{Z}_m}^n$ be two words. The *overlap* of x and y is the intersection of their supports,

$$overlap(x, y) = supp(x) \cap supp(y).$$

Let $A \subseteq \{1, ..., n\}$. We define for $x \in {\mathbb{Z}_m}^n$ the *restriction* $x|A \in {\mathbb{Z}_m}^n$ by

$$x|A(i) = \begin{cases} x(i) & \text{for } i \in A \\ 0 & \text{for } i \notin A. \end{cases}$$

Lemma 3.1. *Let $x, y \in {\mathbb{Z}_m}^n$, $A = overlap(x, y)$. Then we have*

$$d(x, y) = w(x) + w(y) - 2|A| + d(x|A, y|A).$$

Proof. We denote by A' the set of positions which are not in A. Since $A = overlap(x, y)$ we have

$$supp(x|A') \cap supp(y|A') = \phi$$

and

$$d(x|A', y|A') = w(x) - |A| + w(y) - |A| = w(x) + w(y) - 2|A|.$$

Therefore we conclude

$$\begin{aligned} d(x,y) &= d(x|A', y|A') + d(x|A, y|A) \\ &= w(x) + w(y) - 2|A| + d(x|A, y|A). \end{aligned}$$

$\square$

Lemma 3.2. *Let $m, n, r, s, t, \varepsilon$ be integers, $m \geq 2,\ \varepsilon \in \{0,1\},\ n \geq 2t+\varepsilon > 0,\ 1 \leq r \leq t+\varepsilon,\ \ -(r-\varepsilon) \leq s \leq r$. Then for adjacent vertices x, y of $HG(m, n, 2t+\varepsilon),\ w(x) = t+r,\ w(y) = t+s$, the following statements hold.*

1.

$$|overlap(x,y)| \geq \lceil \frac{r+s-\varepsilon}{2} \rceil = O_{min}$$

We call O_{min} the minimal overlap of adjacent words of weight $t+r$ and $t+s$ in $HG(m, n, 2t+\varepsilon)$.

2. *Let*

$$O_{min} = \lceil \frac{r+s-\varepsilon}{2} \rceil = \frac{r+s-\varepsilon+\delta}{2}$$

and

$$\delta = \begin{cases} 0 & \text{if } r+s-\varepsilon \text{ is even} \\ 1 & \text{if } r+s-\varepsilon \text{ is odd .} \end{cases}$$

If $|overlap(x,y)| = O_{min} + p,\ 0 \leq p \leq O_{min} - \delta$, then x and y have at least $O_{min} - (\delta + p)$ positions with the same entries in the $overlap(x,y)$.

Proof.

1. Let $A = overlap(x,y)$. We have by Lemma 3.1

$$d(x,y) = w(x) + w(y) - 2|A| + d(x|A, y|A).$$

As x, y are adjacent we conclude $d(x,y) \leq 2t+\varepsilon$ and

$$w(x) + w(y) - 2|A| \leq d(x,y) \leq 2t+\varepsilon,$$

$$t+r+t+s-2|A| \leq 2t+\varepsilon,\ 2|A| \geq r+s-\varepsilon.$$

For the integer $|A|$ we must have $|A| \geq \lceil \frac{r+s-\varepsilon}{2} \rceil$.

2. Let $A = overlap(x,y)$ and

$$|A| = \mathrm{O_{min}} + p = \frac{r+s-\varepsilon+\delta}{2} + p.$$

By Lemma 3.1 we have

$$\begin{aligned} d(x,y) &= t+r+t+s-2|A|+d(x|A,y|A) \\ &= 2t+r+s-(r+s-\varepsilon+\delta)-2p+d(x|A,y|A) \\ &= 2t+\varepsilon-\delta-2p+d(x|A,y|A). \end{aligned}$$

As $d(x,y) \leq 2t+\varepsilon$ we conclude

$$2t+\varepsilon-\delta-2p+d(x|A,y|A) \leq 2t+\varepsilon,$$

$$d(x|A,y|A) \leq 2p+\delta.$$

The number of common entries of x and y in A is

$$\begin{aligned} |A|-d(x|A,y|A) &\geq |A|-2p-\delta \\ &\geq \mathrm{O}_{\min}+p-2p-\delta = \mathrm{O}_{\min}-(p+\delta). \end{aligned}$$

□

Theorem 3.1. *For integers $m,t,\ m \geq 2,\ t \geq 1$, there is a positive integer n_0 (depending only on t and m) such that for every integer $n \geq n_0$ the following statement is true:*
Suppose C is a maximal clique of $HG(m,n,2t)$, $\overline{0} \in C$ and $t+r$, $1 \leq r \leq t$, is the maximum weight of a word in C. Then there are positions $i_1,...,i_r$, $1 \leq i_1 < ... < i_r \leq n$, nonzero integers $a_{i_1},...,a_{i_r} \in \mathbb{Z}_m$ and $2t+1$ words $u^{(1)},...,u^{(2t+1)}$ in C of weight $t+r$ satisfying

$$u^{(j)}(i_1)=a_{i_1},...,u^{(j)}(i_r)=a_{i_r}$$

for every j, $1 \leq j \leq 2t+1$, and the supports of any two of these words overlap exactly in positions $i_1,...,i_r$.

Proof. For every $n \geq 2t$ we know by Proposition 1.2

$$|C| \ \geq \ \sum_{l=0}^{t} \binom{n}{l}(m-1)^l \ \geq \ \binom{n}{t}(m-1)^t \ \geq \ \binom{n}{t}.$$

We estimate the binomial coefficient:

$$\begin{aligned} \binom{n}{t} &= \frac{n(n-1)...(n-(t-1))}{t!} = \frac{1}{t!}(1-\frac{1}{n})...(1-\frac{t-1}{n})n^t \\ &\geq \frac{1}{t!}(1-\frac{t-1}{n})^{t-1}n^t. \end{aligned}$$

By Bernoulli's equation, $(1-\alpha)^n \geq 1 - n\alpha$, we conclude

$$\binom{n}{t} \geq \frac{1}{t!}(1 - \frac{(t-1)^2}{n})n^t.$$

For $n \geq 2(t-1)^2$ we have

$$1 - \frac{(t-1)^2}{n} \geq \frac{1}{2}$$

and therefore

$$|C| \geq \sum_{l=0}^{t} \binom{n}{l} \geq c_1 n^t$$

with $c_1 = \frac{1}{2t!}$.

Let $u^{(1)} \in C$ be an arbitrary word of maximum weight $t+r$. Without loss of generality(w.l.o.g) we may assume

$$u^{(1)}(1) = u^{(1)}(2) = ... = u^{(1)}(t+r) = 1.$$

Let A_1 be the number of words with weight up to $t-r$,

$$A_1 = \sum_{l=0}^{t-r} \binom{n}{l}(m-1)^l.$$

As these words have distance at most $2t$ to every word of C, the maximal clique C must contain all of these words.

Suppose that A'_s is the number of words of weight $t+s$, $-(r-1) \leq s \leq r-1$, in C. Let A_2 be the number of words of weights $t-r+1, ..., t+r-1$ in C,

$$A_2 = \sum_{s=-r+1}^{r-1} A'_s.$$

By Lemma 3.2(1), for every word v of weight $t+s$ $(-r+1 \leq s \leq r-1)$ in C we have $|overlap(v, u^{(1)})| \geq \lceil \frac{r+s}{2} \rceil$, which implies

$$A'_s \leq \sum_{k=\lceil \frac{r+s}{2} \rceil}^{t+s} \binom{t+r}{k}\binom{n-(t+r)}{t+s-k}(m-1)^{t+s}. \qquad (3.1)$$

Transformation of the sum by Lemma 2.1(1) yields

$$A'_s \leq \left[\binom{n}{t+s} - \sum_{k=0}^{\lceil \frac{r+s}{2} \rceil - 1} \binom{t+r}{k}\binom{n-(t+r)}{t+s-k}\right](m-1)^{t+s}. \qquad (3.2)$$

Let B_1 be the number of words of weight at most $t+r-1$ in C,

$$B_1 = A_1 + A_2 = A_1 + \sum_{s=-r+1}^{-1} A'_s + \sum_{s=0}^{r-1} A'_s.$$

Applying (3.2) for $-r+1 \leq s < 0$, yields

$$B_1 \;\leq\; \sum_{l=0}^{t-1} \binom{n}{l}(m-1)^l + \sum_{s=0}^{r-1} A'_s. \tag{3.3}$$

Observing

$$\binom{n}{l} \;=\; \frac{n(n-1)...(n-l+1)}{l!} \;\leq\; n^l$$

we deduce from (3.1) for $0 \leq s \leq r-1$:

$$\begin{aligned}
A'_s \quad &\leq \quad \sum_{k=\lceil\frac{r+s}{2}\rceil}^{t+s} \binom{t+r}{k}\binom{n-(t+r)}{t+s-k}(m-1)^{t+s} \\
&\leq \quad \sum_{k=\lceil\frac{r+s}{2}\rceil}^{t+s} (t+r)^k(n-(t+r))^{t+s-k}(m-1)^{t+s} \\
&\leq \quad \sum_{k=\lceil\frac{r+s}{2}\rceil}^{t+s} (t+r)^k n^{t+s-k}(m-1)^{t+s} \\
&\leq \quad (t+s-\lceil\frac{r+s}{2}\rceil+1)(t+r)^{t+s}n^{t+s-\lceil\frac{r+s}{2}\rceil}(m-1)^{t+s}.
\end{aligned}$$

But $0 \leq s \leq r-1$ implies $s+1 = \lceil\frac{2s+1}{2}\rceil \leq \lceil\frac{r+s}{2}\rceil$ and

$$t+s-\lceil\frac{r+s}{2}\rceil+1 \;\leq\; t+s-(s+1)+1 = t.$$

Therefore we have

$$A'_s \;\leq\; t(t+r)^{t+s}(m-1)^{t+s}n^{t-1}.$$

Because of $1 \leq r \leq t$ we see

$$\sum_{s=0}^{r-1} A'_s \;\leq\; rt(t+r)^{t+r-1}(m-1)^{t+r-1}n^{t-1} \;\leq\; t^{2t+1}2^{2t-1}(m-1)^{2t-1}n^{t-1}.$$

Inserting this estimate and

$$\sum_{l=0}^{t-1} \binom{n}{l}(m-1)^l \;\leq\; \sum_{l=0}^{t-1} n^l(m-1)^l \;\leq\; tn^{t-1}(m-1)^{t-1}$$

in equation (3.3) yields

$$B_1 \;\leq\; (t(m-1)^{t-1} + t^{2t+1}2^{2t-1}(m-1)^{2t-1})n^{t-1}. \qquad (3.4)$$

Therefore there are $d_1 \in \mathbb{R},\ n_1 \in \mathbb{N},\ 0 < d_1 < c_1$, depending only on m and t such that

$$B_1 \;\leq\; d_1 n^t \text{ for every } n \geq n_1.$$

Let B_2 be the number of words of weight $t+r$ in C then

$$B_2 \;\geq\; (c_1 - d_1)n^t \;=\; c_2 n^t \text{ for } n \geq n_1.$$

Define B_3 as the number of words of weight $t+r$ in C, which intersect with $u^{(1)}$ in more than r positions. We have

$$\begin{aligned}
B_3 &\leq \sum_{k=r+1}^{t+r} \binom{t+r}{k}\binom{n-(t+r)}{t+r-k}(m-1)^{t+r} \\
&\leq \sum_{k=r+1}^{t+r} (t+r)^k (n-(t+r))^{t+r-k}(m-1)^{t+r} \\
&\leq \sum_{k=r+1}^{t+r} (t+r)^{t+r} n^{t+r-k}(m-1)^{t+r} \\
&\leq t(t+r)^{t+r}(m-1)^{t+r}n^{t-1} \;\leq\; 2^{2t}t^{2t+1}(m-1)^{2t}n^{t-1},
\end{aligned}$$

where we applied $1 \leq r \leq t$ for the last inequality. Thus there are $d_2 \in \mathbb{R},\ n_2 \in \mathbb{N},\ 0 < d_2 < c_2$, depending only on m and t such that

$$B_3 \leq d_2 n^t \text{ for } n \geq n_2.$$

According to Lemma 3.2(1) every word of weight $t+r$ in C has minimal overlap r with $u^{(1)}$. So the number of words of weight $t+r$ in C, which intersect with $u^{(1)}$ in exactly r positions is

$$B_4 \;\geq\; (c_2 - d_2)n^t \;=\; c_3 n^t \text{ for } n \geq n_2.$$

By Lemma 3.2(2) these words coincide in exactly r positions with $u^{(1)}$, which means that we have entries 1 in these positions.
The common entries 1 of these words with $u^{(1)}$ are distributed among

$$\binom{t+r}{r} \leq (t+r)^r \leq (2t)^t = h$$

subsets of r positions from $1, ..., t+r$. There is a subset of r positions (w.l.o.g. 1,...,r) such that there are at least $\frac{c_3}{h}n^t = c_4 n^t$ of the at last considered words of weigh $t+r$, which have common entries 1 with $u^{(1)}$ exactly in positions $1,...,r$.

Let $u^{(2)}$ be one of these words. Thus $u^{(1)}$ and $u^{(2)}$ overlap with entries 1 exactly in positions $1,...,r$. Among the at last considered $c_4 n^t$ words, the number of words,which meet $u^{(2)}$ in more than r entries 1, is

$$B_5 \;\leq\; \sum_{k=1}^{t} \binom{t}{k}\binom{n-(2t+r)}{t-k}(m-1)^t \;\leq\; t^{t+1}(m-1)^t n^{t-1}. \qquad (3.5)$$

There are $d_3 \in \mathbb{R},\ n_3 \in \mathbb{N},\ 0 < d_3 < c_4$, depending only on m and t such that

$$B_5 \leq d_3 n^t \text{ for } n \geq n_3.$$

Among the at last considered $c_4 n^t$ words, the number of words, which meet $u^{(1)}, u^{(2)}$ in exactly r enteries 1 (namely in positions 1,...,r) is

$$B_6 \;\geq\; (c_4 - d_3)n^t \;=\; c_5 n^t \text{ for } n \geq n_3.$$

Let $u^{(3)}$ be one of these words.

We repeat this process until for $n \geq n_{2t+1} = n_0$ we have found $2t+1$ words

$$u^{(1)}, u^{(2)}, ..., u^{(2t+1)},$$

which intersect with entries 1 exactly in positions 1,...,r and have pairwise disjoint supports in all positions $i > r$. $\square$

Corollary 3.1. *Let m, t be integers, $m \geq 2$, $t \geq 1$, and n_0 be determined as in Theorem 3.1, $n \geq n_0$. Suppose $t+r$ $(1 \leq r \leq t)$ is the maximum weight of a word in a maximal clique C of $HG(m,n,2t)$, $\overline{0} \in C$. Let the words $u^{(1)},...,u^{(2t+1)}$, the positions $i_1,...,i_r$ and nonzero integers $a_{i_1},...,a_{i_r}$ be determined for $n \geq n_0$ according to Theorem 3.1. Then every word $v \in C$ of weight $t+s$,$-r+1 \leq s \leq r$, must have*

$$|supp(v) \cap \{i_1,...,i_r\}| \geq \lceil \frac{r+s}{2} \rceil.$$

Also for every word $v \in C$ of weight $t+r$ we have

$$v(i_1) = a_{i_1}, ..., v(i_r) = a_{i_r}.$$

Proof. W.l.o.g. we may assume $i_1 = 1, ..., i_r = r$ and $a_{i_1} = ... = a_{i_r} = 1$. The words $u^{(1)},...,u^{(2t+1)}$ have maximum weight $t+r$ such that

$$overlap(u^{(i)}, u^{(j)}) = \{1,...,r\}$$

for every $i, j, i \neq j,\ 1 \leq i, j \leq 2t+1$.
Let $v \in C,\ w(v) = t+s,\ \ -r+1 \leq s \leq r$. By Lemma 3.2(1), we have

$$|overlap(u^{(i)}, v)| \geq \lceil \frac{r+s}{2} \rceil = k$$

for every $i,\ 1 \leq i \leq 2t+1$. If $|supp(v) \cap \{1, ..., r\}| < k$ then $supp(v)$ must have an additional common position with the support of every word $u^{(i)},\ 1 \leq i \leq 2t+1$. But this would raise the weight of v at least up to $2t+1$, which is a contradiction.

Let $v \in C$, $w(v) = t+r$. Then by part 1 we have

$$\{1, ..., r\} \subseteq supp(v).$$

Suppose that there is $l \in \{1, ..., r\}$ such that $v(l) \neq 1$. By Lemma 3.1 with $A = overlap(v, u^{(i)})$ we have

$$d(v, u^{(i)}) = w(v) + w(u^{(i)}) - 2|A| + d(v|A, u^{(i)}|A) \leq 2t,$$

$$2(t+r) - 2|A| + 1 \leq 2t,$$

$$|A| \geq \lceil \frac{2r+1}{2} \rceil = r+1.$$

So $supp(v)$ must have an additional common position $i > r$ with the support of every word $u^{(i)}$. Again this would raise the weight of v above $2t$. □

Corollary 3.2. *Let m, t be integers, $m \geq 2$, $t \geq 1$, and n_0 be determined as in Theorem 3.1, $n \geq n_0$. Suppose $t+r$ $(1 \leq r \leq t)$ is the maximum weight of a word in a maximal clique C of $HG(m, n, 2t)$, $\overline{0} \in C$. Let the words $u^{(1)}, ..., u^{(2t+1)}$, the positions $i_1, ..., i_r$ and nonzero integers $a_{i_1}, ..., a_{i_r}$ be determined for $n \geq n_0$ according to Theorem 3.1. Suppose $v \in C$ has weight $t+s$,$-r+1 \leq s \leq r$, and*

$$|supp(v) \cap \{i_1, ..., i_r\}| = \lceil \frac{r+s}{2} \rceil + k, \quad 0 \leq k \leq r - \lceil \frac{r+s}{2} \rceil.$$

Denote by X the number of positions $i_j \in \{i_1, ..., i_r\} \cap supp(v)$ with $v(i_j) \neq a_{i_j}$. Then we have

$$X \leq 2\lceil \frac{r+s}{2} \rceil - (r+s) + 2k.$$

Proof. Let w.l.o.g. $i_1 = 1, ..., i_r = r$, $B = \{1, ..., r\}$ and $|supp(v) \cap B| = \lceil \frac{r+s}{2} \rceil + k$. For every i, $1 \leq i \leq 2t+1$, we have

$$d(v|B, u^{(i)}|B) = r - \lceil \frac{r+s}{2} \rceil - k + X.$$

Let $B' = \{1, ..., n\} \setminus B$. As $w(v) < 2t+1$, there exists i, $1 \leq i \leq 2t+1$, such that

$$supp(v|B') \cap supp(u^{(i)}|B') = \phi.$$

Then we have

$$d(v|B', u^{(i)}|B') = t + t + s - \lceil \frac{r+s}{2} \rceil - k,$$

thus

$$d(v, u^{(i)}) = d(v|B, u^{(i)}|B) + d(v|B', u^{(i)}|B') = 2t + r + s - 2\lceil \frac{r+s}{2} \rceil - 2k + X.$$

But v and $u^{(i)}$ are in C, therefore

$$d(v, u^{(i)}) = 2t + r + s - 2\lceil \frac{r+s}{2} \rceil - 2k + X \leq 2t,$$

$$X \leq 2\lceil \frac{r+s}{2} \rceil - (r+s) + 2k.$$

□

The ω_0-conjecture for even distance at most 6 will be a consequence of the following theorem.

Theorem 3.2. *For integers m, t, $m \geq 2$, $t \geq 1$, there is a positive integer n_0 (depending only on t and m) such that for every integer $n \geq n_0$ the following statement is true:*
Suppose C is a maximal clique of $HG(m, n, 2t)$, $\overline{0} \in C$ and $t + r$, $0 \leq r \leq 3$, is the maximum weight of a word in C, then

$$|C| = \sum_{j=0}^{t} \binom{n}{j} (m-1)^j.$$

Proof. Suppose that C is a maximal clique of $HG(m, n, 2t)$, $\overline{0} \in C$ and $t + r$ is the maximum weight of a word in C, $0 \leq r \leq 3$. For every $n \geq 2t$ we know by Proposition 1.2

$$|C| \geq \sum_{j=0}^{t} \binom{n}{j} (m-1)^j.$$

If $r = 0$ then

$$|C| = \sum_{j=0}^{t} \binom{n}{j} (m-1)^j.$$

Therefore we consider $1 \leq r \leq 3$. Let n_0 be chosen according to Theorem 3.1. By Theorem 3.1 and Corollary 3.1 there are positions $i_1, ..., i_r$ and nonzero integers $a_{i_1}, ..., a_{i_r}$, w.l.o.g. $i_1 = 1, ..., i_r = r$ and $a_{i_j} = 1$ for $1 \leq j \leq r$, such that for every word $u \in C$ of weight $t + r$ we have

$$u(1) = u(2) = ... = u(r) = 1.$$

Let A_1 be the number of all words in C of weight at most $t-r$. These words have at most distance $2t$ to every word in C. They must be contained in C, because C is maximal. We have

$$A_1 = \sum_{j=0}^{t-r} \binom{n}{j} (m-1)^j.$$

If we denote by A_2 the number of all words in C of weight at least $t-r+1$ then we have

$$|C| = A_1 + A_2. \tag{3.6}$$

Let A'_s be the number of all words in C of weight $t+s$, $-r+1 \leq s \leq r$. Then we have

$$A_2 = \sum_{s=-r+1}^{r} A'_s. \tag{3.7}$$

Consider the following cases.

Case 1: $r = 1$
We have

$$A_1 = \sum_{j=0}^{t-1} \binom{n}{j} (m-1)^j.$$

If $0 \leq s \leq 1$ then $\lceil \frac{r+s}{2} \rceil = 1$. Corollary 3.1 yields

$$A'_1 \leq \binom{1}{0} \binom{n-1}{t} (m-1)^t.$$

Corollary 3.2 with $k = 0$ and $X \leq 1$ implies

$$A'_0 \leq \binom{1}{1} \binom{n-1}{t-1} (m-1)^t.$$

Therefore we have by (3.7)

$$A_2 \leq \binom{1}{0} \binom{n-1}{t} (m-1)^t + \binom{1}{1} \binom{n-1}{t-1} (m-1)^t$$

and by Lemma 2.1(1)

$$A_2 \leq \binom{n}{t}(m-1)^t,$$

thus

$$|C| = A_1 + A_2 \leq \sum_{j=0}^{t} \binom{n}{j}(m-1)^j.$$

Case 2: $r = 2$
We have $r = 2 \leq t$ and

$$A_1 = \sum_{j=0}^{t-2} \binom{n}{j}(m-1)^j.$$

Consider $-1 \leq s \leq 2$. For $s = 1, 2$ we have $\lceil \frac{r+s}{2} \rceil = 2$. Corollary 3.1 yields

$$A_2' \leq \binom{2}{0}\binom{n-2}{t}(m-1)^t.$$

By Corollary 3.2 with $k = 0$ and $X \leq 1$ we deduce

$$A_1' \leq \binom{2}{1}\binom{n-2}{t-1}(m-1)^t - \binom{n-2}{t-1}(m-1)^{t-1}.$$

For $s = 0, -1$ we get $\lceil \frac{r+s}{2} \rceil = 1$. In the notation of Corollary 3.2 we have $0 \leq k \leq 1$ and $X \leq 2k$ for $s = 0$ and $X \leq 1 + 2k$ for $s = -1$. Thus we conclude

$$A_0' \leq \binom{2}{1}\binom{n-2}{t-1}(m-1)^{t-1} + \binom{2}{2}\binom{n-2}{t-2}(m-1)^t$$

and

$$A_{-1}' \leq \binom{2}{1}\binom{n-2}{t-2}(m-1)^{t-1} + \binom{2}{2}\binom{n-2}{t-3}(m-1)^{t-1}.$$

Therefore we have by (3.7)

$$\begin{aligned} A_2 \quad \leq \quad & \left(\binom{2}{0}\binom{n-2}{t} + \binom{2}{1}\binom{n-2}{t-1} + \binom{2}{2}\binom{n-2}{t-2}\right)(m-1)^t \\ & + \left(\binom{n-2}{t-1} + \binom{2}{1}\binom{n-2}{t-2} + \binom{2}{2}\binom{n-2}{t-3}\right)(m-1)^{t-1} \end{aligned}$$

and by Lemma 2.1(1)

$$A_2 \leq \binom{n}{t}(m-1)^t + \binom{n}{t-1}(m-1)^{t-1}.$$

Finally we get

$$
\begin{aligned}
|C| = A_1 + A_2 &\leq \sum_{j=0}^{t-2} \binom{n}{j}(m-1)^j + \binom{n}{t}(m-1)^t + \binom{n}{t-1}(m-1)^{t-1} \\
&\leq \sum_{j=0}^{t} \binom{n}{j}(m-1)^j.
\end{aligned}
$$

Case 3: $r = 3$
We have $r = 3 \leq t$ and

$$A_1 = \sum_{j=0}^{t-3} \binom{n}{j}(m-1)^j.$$

Consider $-2 \leq s \leq 3$. For $s = 2, 3$ we have $\lceil \frac{r+s}{2} \rceil = 3$. Corollary 3.1 yields

$$A'_3 \leq \binom{3}{0}\binom{n-3}{t}(m-1)^t.$$

By Corollary 3.2 with $k = 0$ and $X \leq 1$ we deduce

$$
\begin{aligned}
A'_2 &\leq \binom{3}{1}\binom{n-3}{t-1}(m-1)^t - 3\binom{n-3}{t-1}(m-1)^{t-1} + \binom{n-3}{t-1}(m-1)^{t-1} \\
&\leq \binom{3}{1}\binom{n-3}{t-1}(m-1)^t - 2\binom{n-3}{t-1}(m-1)^{t-1}.
\end{aligned}
$$

For $s = 1, 0$ we see $\lceil \frac{r+s}{2} \rceil = 2$. In the notation of Corollary 3.2 we have $0 \leq k \leq 1$ and $X \leq 2k$ for $s = 1$ and $X \leq 1+2k$ for $s = 0$. Thus we conclude

$$
\begin{aligned}
A'_1 \leq\ & \binom{3}{2}\binom{n-3}{t-1}(m-1)^{t-1} + \binom{3}{2}\binom{n-3}{t-2}(m-1)^t \\
& -3\binom{n-3}{t-2}(m-1)^{t-1} + \binom{n-3}{t-2}(m-1)^{t-2},
\end{aligned}
$$

therefore

$$A'_1 \leq \binom{3}{2}\binom{n-3}{t-2}(m-1)^t + \left[\binom{3}{2}\binom{n-3}{t-1} - 3\binom{n-3}{t-2}\right](m-1)^{t-1} +$$

$$\binom{n-3}{t-2}(m-1)^{t-2}.$$

Similarly we see

$$A'_0 \leq \binom{2}{1}\binom{3}{2}\binom{n-3}{t-2}(m-1)^{t-1} - 3\binom{n-3}{t-2}(m-1)^{t-2} + \binom{3}{3}\binom{n-3}{t-3}(m-1)^{t}.$$

For $s = -1, -2$ we see $\lceil \frac{r+s}{2} \rceil = 1$. In the notation of Corollary 3.2 we have $0 \leq k \leq 2$ and respectively $X \leq 2k$ for $s = -1$ and $X \leq 1 + 2k$ for $s = -2$. Thus we conclude

$$\begin{aligned} A'_{-1} &\leq \binom{3}{1}\binom{n-3}{t-2}(m-1)^{t-2} + \binom{3}{2}\binom{n-3}{t-3}(m-1)^{t-1} + \\ &\quad \binom{3}{3}\binom{n-3}{t-4}(m-1)^{t-1} \\ &\leq \left[\binom{3}{2}\binom{n-3}{t-3} + \binom{3}{3}\binom{n-3}{t-4}\right](m-1)^{t-1} + \binom{3}{1}\binom{n-3}{t-2}(m-1)^{t-2} \end{aligned}$$

and

$$A'_{-2} \leq \binom{3}{1}\binom{n-3}{t-3}(m-1)^{t-2} + \binom{3}{2}\binom{n-3}{t-4}(m-1)^{t-2} + \binom{3}{3}\binom{n-3}{t-5}(m-1)^{t-2}.$$

Therefore equation (3.7) yields

$$A_2 \leq$$

$$\left[\binom{3}{0}\binom{n-3}{t} + \binom{3}{1}\binom{n-3}{t-1} + \binom{3}{2}\binom{n-3}{t-2} + \binom{3}{3}\binom{n-3}{t-3}\right](m-1)^{t} +$$
$$\left[\binom{3}{0}\binom{n-3}{t-1} + \binom{3}{1}\binom{n-3}{t-2} + \binom{3}{2}\binom{n-3}{t-3} + \binom{3}{3}\binom{n-3}{t-4}\right](m-1)^{t-1} +$$
$$\left[\binom{3}{0}\binom{n-3}{t-2} + \binom{3}{1}\binom{n-3}{t-3} + \binom{3}{2}\binom{n-3}{t-4} + \binom{3}{3}\binom{n-3}{t-5}\right](m-1)^{t-2}.$$

This reduces by Lemma 2.1(1) to

$$A_2 \leq \binom{n}{t}(m-1)^{t} + \binom{n}{t-1}(m-1)^{t-1} + \binom{n}{t-2}(m-1)^{t-2},$$

which confirms

$$|C| = A_1 + A_2 \leq \sum_{j=0}^{t}\binom{n}{j}(m-1)^{j}.$$

□

Corollary 3.3. *For integers $m, t,\ m \geq 2,\ t \in \{1, 2, 3\}$, there is $n_0 \in \mathbb{N}$ (depending only on t and m) such that for every $n \geq n_0$*

$$\omega(HG(m, n, 2t)) = \sum_{j=0}^{t} \binom{n}{j} (m-1)^j.$$

Proof. Choose n_0 according to Theorem 3.2 and let $n \geq n_0$. Suppose C is a maximal clique in $HG(m, n, 2t)$. By Proposition 1.1, we may assume $\overline{0} \in C$. If $t + r$ is the maximal weight of a word in C, then we have

$$t + r \leq 2t,\ \ r \leq t \leq 3.$$

Theorem 3.2 implies

$$\omega(HG(m, n, 2t)) = |C| = \sum_{j=0}^{t} \binom{n}{j} (m-1)^j.$$

$\square$

Chapter 4

Hamming Graphs with Odd Distance

This chapter is dedicated to Hamming graphs $HG(m, n, d)$ with odd distance parameter d and their clique numbers. We show that the ω_0-conjecture is true for Hamming graphs which have odd distance at most 5.

Lemma 4.1. *Let $c > 0$ be a real constant and m, s, t positive integers, $1 \le s \le t + 1$, $m \ge 2$. Suppose that the infinite set $M \subseteq \mathbb{N}$ has the following property. For every $n \in M$ there is a clique C_n in $HG(m, n, 2t + 1)$ such that $\overline{0} \in C_n$ and C_n has at least cn^t words of weight $t + s$.*
Then there is an integer n_0 (depending only on c, m, t) such that for every $n \ge n_0$, $n \in M$, the following statement is true:
There are positions $i_1, ..., i_s$, $1 \le i_1 < ... < i_s \le n$, nonzero integers $a_{i_1}, ..., a_{i_s} \in \mathbb{Z}_m$ and $2t+2$ words $v^{(1)}, ..., v^{(2t+2)}$ of weight $t+s$ in C_n satisfying

$$v^{(j)}(i_1) = a_{i_1}, ..., v^{(j)}(i_s) = a_{i_s}$$

for every j, $1 \le j \le 2t + 2$.
The supports of any two of these words intersect exactly in positions $i_1, ..., i_s$.
For every word $v \in C_n$ of weight $t + s$ holds

$$v(i_1) = a_{i_1}, ..., v(i_s) = a_{i_s}$$

except possibly for one position $i_j \in \{i_1, ..., i_s\}$, where we may have a nonzero entry $v(i_j) \neq a_{i_j}$.

Proof. Let $v^{(0)}$ be a word of weight $t + s$ in C_n, $n \in M$, $n \ge 2t + 1$. Without loss of generality (w.l.o.g) we may assume

$$v^{(0)}(1) = v^{(0)}(2) = ... = v^{(0)}(t + s) = 1.$$

By Lemma 3.2(1) for every word u of weight $t+s$ in C_n we have

$$|overlap(u, v^{(0)})| \geq \lceil \frac{s+s-1}{2} \rceil = s.$$

Define B_1 as the number of words u of weight $t+s$ in C_n, such that $|overlap(u, v^{(0)})| > s$. We have

$$\begin{aligned}
B_1 &\leq \sum_{k=s+1}^{t+s} \binom{t+s}{k}\binom{n-(t+s)}{t+s-k}(m-1)^{t+s} \\
&\leq \sum_{k=s+1}^{t+s} (t+s)^k (n-(t+s))^{t+s-k}(m-1)^{t+s} \\
&\leq \sum_{k=s+1}^{t+s} (t+s)^{t+s} n^{t+s-k}(m-1)^{t+s} \\
&\leq t(t+s)^{t+s} n^{t-1}(m-1)^{t+s} \ \leq\ t(2t+1)^{2t+1} n^{t-1}(m-1)^{2t+1},
\end{aligned}$$

where we applied $1 \leq s \leq t+1$ for the last inequality. Thus there are $d_1 \in \mathbb{R},\ n_1 \in \mathbb{N}$ depending only on c, m, t such that

$$0 < d_1 < c \text{ and } B_1 \leq d_1 n^t \text{ for } n \geq n_1.$$

So the number of words u of weight $t+s$ in C_n with $|overlap(u, v^{(0)})| = s$ is

$$B_2 \ \geq\ (c-d_1)n^t \ =\ c_2 n^t \text{ for } n \geq n_1.$$

The common positions in the support of these words with $supp(v^{(0)})$ are distributed among $\binom{t+s}{s} \leq \binom{2t+1}{t+1} = h$ subsets of s positions from $1, ..., t+s$. There is a subset of s positions (w.l.o.g 1,...,s) such that there are at least $\frac{c_2}{h} n^t = c_3 n^t$ of the at last considered words u of weight $t+s$ with $supp(u) \cap supp(v^{(0)}) = \{1, ..., s\}$. By Lemma 3.2(2), these words and $v^{(0)}$ have at least $s-1$ common entries in the positions 1,...,s. Thus there is at most one position among 1,...,s, in which these words may have a nonzero entry not equal to 1. There are at least

$$\frac{c_3}{m^s} n^t \geq \frac{c_3}{m^{t+1}} n^t = c_4 n^t$$

of the at last considered $c_3 n^t$ words of weight $t+s$, which have common entries in positions $1, ..., s$.

Let $v^{(1)}$ be one of these words, w.l.o.g. suppose

$$v^{(1)}(1) = ... = v^{(1)}(s-1) = 1,\ v^{(1)}(s) = a \neq 0.$$

Among the at last considered c_4n^t words, the number of words, which meet $v^{(1)}$ in more than s nonzero entries is

$$B_3 \;\leq\; \sum_{k=1}^{t} \binom{t}{k}\binom{n-(t+s)}{t-k}(m-1)^{t+s} \;\leq\; t^{t+1}n^{t-1}(m-1)^{2t+1}.$$

There are $d_2 \in \mathbb{R},\; n_2 \in \mathbb{N}$ depending only on c, m, t such that

$$0 < d_2 < c_4 \text{ and } B_3 \leq d_2 n^t \text{ for } n \geq n_2.$$

Among the at last considered c_4n^t words the number of words, which meet $v^{(1)}$ in exactly s nonzero enteries (namely in positions 1,...,s), is

$$B_4 \;\geq\; (c_4 - d_2)n^t \;=\; c_5 n^t \text{ for } n \geq n_2.$$

Let $v^{(2)}$ be one of these words. Thus we have $v^{(2)}(i) = v^{(1)}(i)$ for every $i,\; 1 \leq i \leq s$.

We repeat this process until for $n \geq n_0 = n_{2t+2}$ we have found $2t+2$ words

$$v^{(1)}, v^{(2)}, ..., v^{(2t+2)},$$

of weight $t+s$ such that $v^{(i)}(1) = ... = v^{(i)}(s-1) = 1,\; v^{(i)}(s) = a \neq 0$ for every $i,\; 1 \leq i \leq 2t+2$, and such that these words have pairwise disjoint supports in all positions $i > s$.

Let $v \in C_n,\; w(v) = t+s,\, n \geq n_0$. By Lemma 3.2(1), we have

$$|overlap(v, v^{(i)})| \geq s$$

for every $i,\; 1 \leq i \leq 2t+2$. If $|supp(v) \cap \{1, ..., s\}| < s$ then $supp(v)$ must have an additional common position with the support of every word $v^{(i)},\; 1 \leq i \leq 2t+2$. But this would raise the weight of v at least up to $2t+2$, which is a contradiction. Therefore $\{1, ..., s\} \subseteq supp(v)$.

As $w(v) < 2t+2$, $supp(v)$ can not meet every $supp(v^{(i)})$, $1 \leq i \leq 2t+2$, in a position $k_i > s$. Therefore there is a word $v^{(i)}$ with

$$supp(v) \cap supp(v^{(i)}) = \{1, ..., s\}.$$

Now by Lemma 3.3(2) words v and $v^{(i)}$ have the same entries in positions $1, ..., s$ except possibly for one position $i_j \in \{1, ..., s\}$. $\square$

Theorem 4.1. *For every pair of positive integers m, t, $m \geq 2$, there is an integer n_0 (depending only on m, t) such that for every integer $n \geq n_0$ the following statement is true:*
Suppose C is a maximal clique of $HG(m, n, 2t+1)$, $\overline{0} \in C$ and $t+r$, $1 \leq r \leq t+1$, is the maximum weight of a word in C. Then there are positions $i_1, ..., i_r$, $1 \leq i_1 < ... < i_r \leq n$, nonzero integers $a_{i_1}, ..., a_{i_r} \in \mathbb{Z}_m$ and $2t+2$ words $u^{(1)}, ..., u^{(2t+2)}$ in C of weight $t+r$ such that

$$u^{(j)}(i_1) = a_{i_1}, ..., u^{(j)}(i_r) = a_{i_r}$$

for every j, $1 \leq j \leq 2t+2$ and the supports of these words intersect exactly in positions $i_1, ..., i_r$.
Moreover, for $r > 1$ there are positions $k_1, ..., k_{r-1}$, $1 \leq k_1 < ... < k_{r-1}$, $\{k_1, ..., k_{r-1}\} \subset \{i_1, ..., i_r\}$, and $2t+2$ words $v^{(1)}, ..., v^{(2t+2)}$ in C of weight $t+r-1$ such that

$$v^{(j)}(k_1) = a_{k_1}, ..., v^{(j)}(k_{r-1}) = a_{k_{r-1}}$$

for every j, $1 \leq j \leq 2t+2$ and the supports of these words intersect exactly in positions $k_1, ..., k_{r-1}$.

Proof. By Proposition 1.2 we have for $n \geq 2t+1$

$$\begin{aligned} |C| &\geq \sum_{l=0}^{t} \binom{n}{l}(m-1)^l + \binom{n-1}{t}(m-1)^{t+1} \\ &\geq \binom{n}{t}(m-1)^t + \binom{n-1}{t}(m-1)^{t+1}. \end{aligned}$$

We estimate the binomial coefficients:

$$\begin{aligned} \binom{n}{t} &= \frac{n(n-1)...(n-(t-1))}{t!} = \frac{1}{t!}(1-\frac{1}{n})...(1-\frac{t-1}{n})n^t \\ &\geq \frac{1}{t!}(1-\frac{t-1}{n})^{t-1}n^t \end{aligned}$$

and

$$\begin{aligned} \binom{n-1}{t} &= \frac{(n-1)(n-2)...(n-1-(t-1))}{t!} = \frac{1}{t!}(1-\frac{1}{n})...(1-\frac{t}{n})n^t \\ &\geq \frac{1}{t!}(1-\frac{t}{n})^t n^t. \end{aligned}$$

By Bernoulli's equation, $(1-\alpha)^n \geq 1 - n\alpha$, we conclude

$$\binom{n}{t} \geq \frac{1}{t!}(1 - \frac{(t-1)^2}{n})n^t, \ \binom{n-1}{t} \geq \frac{1}{t!}(1 - \frac{t^2}{n})n^t.$$

For $n \geq m^2(t^2+1)$ we have

$$0 \leq \frac{(t-1)^2}{n} \leq \frac{1}{m^2}, \ 0 \leq \frac{t^2}{n} \leq \frac{1}{m^2},$$

$$\binom{n}{t} \geq \frac{1}{m^2 t!}(m^2-1)n^t, \ \binom{n-1}{t} \geq \frac{1}{m^2 t!}(m^2-1)n^t.$$

Now we obtain

$$|C| \geq \binom{n}{t}(m-1)^t + \binom{n-1}{t}(m-1)^{t+1}$$

$$\geq$$

$$\frac{1}{m^2 t!}(m^2-1)(m-1)^t n^t + \frac{1}{m^2 t!}(m^2-1)(m-1)^{t+1} n^t$$

$$=$$

$$\frac{1}{m^2 t!}(m^2-1)(m-1)^t n^t (1+m-1) = \frac{1}{t! m}(m+1)(m-1)^{t+1} n^t.$$

Therefore for $n \geq m^2(t^2+1)$, we have

$$|C| \geq c_1 n^t \tag{4.1}$$

with $c_1 = \frac{1}{t!m}(m+1)(m-1)^{t+1}$.

Let $u^{(0)} \in C$ be an arbitrary word of maximum weight $t+r$, $1 \leq r \leq t+1$. Without loss of generality (w.l.o.g) we may assume

$$u^{(0)}(1) = u^{(0)}(2) = ... = u^{(0)}(t+r) = 1.$$

Let A_1 be the number of words with weight up to $t-r+1$ in $HG(m, n, 2t+1)$,

$$A_1 \ = \ \sum_{l=0}^{t-r+1} \binom{n}{l}(m-1)^l.$$

As these words have distance at most $2t+1$ to every word of C, the maximal clique C must contain all of these words.

Define A'_s for $r \geq 2$ as the number of words of weight $t+s$, $-(r-2) \leq s \leq r-1$, in C. By Lemma 3.2(1), for every word v of weight $t+s$ in C, we have $|overlap(v, u^{(0)})| \geq \lceil \frac{r+s-1}{2} \rceil \geq 1$, which implies

$$A'_s \;\leq\; \sum_{k=\lceil \frac{r+s-1}{2} \rceil}^{t+s} \binom{t+r}{k}\binom{n-(t+r)}{t+s-k}(m-1)^{t+s}. \tag{4.2}$$

Transformation of the sum by Lemma 2.1(1) yields

$$A'_s \;\leq\; \binom{n}{t+s}(m-1)^{t+s} - \sum_{k=0}^{\lceil \frac{r+s-1}{2} \rceil - 1} \binom{t+r}{k}\binom{n-(t+r)}{t+s-k}(m-1)^{t+s}. \tag{4.3}$$

Let B_1 be the number of words of weight up to $t+r-2$ in C. For $r \geq 2$ we have

$$\begin{aligned} B_1 &= A_1 + \sum_{s=-r+2}^{r-2} A'_s \\ B_1 &= \sum_{l=0}^{t-r+1} \binom{n}{l}(m-1)^l + \sum_{s=-r+2}^{-1} A'_s + \sum_{s=0}^{r-2} A'_s. \end{aligned}$$

Applying (4.3) for $s < 0$ yields

$$B_1 \;\leq\; \sum_{l=0}^{t-1} \binom{n}{l}(m-1)^l + \sum_{s=0}^{r-2} A'_s. \tag{4.4}$$

This inequality is also true for $r = 1$, because the last sum is empty in this case. For the following estimate of this sum we may assume $r \geq 2$.

Observing

$$\binom{n}{l} \;=\; \frac{n(n-1)...(n-l+1)}{l!} \;\leq\; n^l$$

we deduce from (4.2) for $0 \leq s \leq r-2$:

$$\begin{aligned} A'_s &\leq \sum_{k=\lceil \frac{r+s-1}{2} \rceil}^{t+s} (t+r)^k (n-(t+r))^{t+s-k}(m-1)^{t+s} \\ &\leq \sum_{k=\lceil \frac{r+s-1}{2} \rceil}^{t+s} (t+r)^k n^{t+s-k}(m-1)^{t+s} \\ &\leq (t+s-\lceil \frac{r+s-1}{2} \rceil + 1)(t+r)^{t+s} n^{t+s-\lceil \frac{r+s-1}{2} \rceil}(m-1)^{t+s}. \end{aligned}$$

But $0 \leq s \leq r-2$ implies $s+1 = \lceil \frac{2s+1}{2} \rceil \leq \lceil \frac{r+s-1}{2} \rceil$ and

$$t+s-\lceil \frac{r+s-1}{2} \rceil + 1 \ \leq \ t+s-(s+1)+1 = t.$$

Therefore we have

$$A'_s \ \leq \ t(t+r)^{t+s} n^{t-1} (m-1)^{t+s}$$

and because of $1 \leq r \leq t+1$ we see

$$\sum_{s=0}^{r-2} A'_s \ \leq \sum_{s=0}^{r-2} t(t+r)^{t+s} n^{t-1} (m-1)^{t+s} \ \leq \ t^2 (2t+1)^{2t-1} n^{t-1} (m-1)^{2t-1}.$$

Inserting this estimate and

$$\sum_{l=0}^{t-1} \binom{n}{l} (m-1)^l \ \leq \ \sum_{l=0}^{t-1} n^l (m-1)^l \ \leq \ t n^{t-1} (m-1)^{t-1}$$

in inequality (4.4) yields

$$\begin{aligned} B_1 \ &\leq \ (t(m-1)^{t-1} + t^2 (2t+1)^{2t-1} (m-1)^{2t-1}) n^{t-1} \\ &\leq \ t^2 (2t+1)^{2t} (m-1)^{2t-1} n^{t-1}. \end{aligned} \tag{4.5}$$

Suppose that

$$n \geq n_1 = m^2 (t^2+1) + m^2 t! t^2 (2t+1)^{2t} (m-1)^{t-2}$$

then $|C| \geq c_1 n^t$ by (4.1). Moreover we have by (4.5)

$$B_1 \ \leq \ d_1 n^t \text{ for } n \geq n_1, \ d_1 = \frac{(m-1)^{t+1}}{m^2 t!}. \tag{4.6}$$

Let B_2 be the number of words of weight $t+r$ or $t+r-1$ in C then

$$B_2 \geq c_1 n^t - d_1 n^t = (c_1 - d_1) n^t = c_2 n^t \tag{4.7}$$

for $n \geq n_1, \ c_2 = \frac{(m-1)^{t+1}(m^2+m-1)}{m^2 t!}$.

Define M as the set of positive integers n, $n \geq n_1$, such that there is a maximal clique C_n in $HG(m,n,2t+1)$, $\overline{0} \in C_n$, which has less than $\frac{1}{2(m^2+m-1)} c_2 n^t$ words of maximum weight $t+r_n$, $1 \leq r_n \leq t+1$. Suppose M is infinite. We show that this assumption leads to a contradiction.
As r_n is restricted, $1 \leq r_n \leq t+1$, we have $r_n = r$ for some fixed r,

$1 \leq r \leq t+1$, infinitely often. So we may assume $r_n = r$ for every $n \in M$. Let B_3 be the number of words of weight $t+r-1$ in C_n. As the number of words of weight $t+r$ in C_n, $n \in M$ is less than $\frac{1}{2(m^2+m-1)} c_2 n^t$, we conclude by (4.7)

$$\begin{aligned} B_3 &\geq \frac{2(m^2+m-1)-1}{2(m^2+m-1)} c_2 n^t \\ &\geq \frac{2(m^2+m-1)-1}{2m^2 t!} (m-1)^{t+1} n^t. \end{aligned} \tag{4.8}$$

If $r = 1$ then we have

$$B_3 \leq \binom{n}{t} (m-1)^t$$

and we conclude by (4.8)

$$\frac{2(m^2+m-1)-1}{2m^2 t!} (m-1)^{t+1} n^t \leq \binom{n}{t} (m-1)^t \leq \frac{n^t}{t!} (m-1)^{t+1}$$

$$\frac{2(m^2+m-1)-1}{2m^2} \leq 1$$

which is contradiction for $m \geq 2$.
For $r > 1$ we apply Lemma 4.1 with $s = r-1$. Then there is an integer n_0' (depending only on m, t) such that for every $n \geq n_0'$, $n \in M$, we have the following property:
There are positions $i_1, ..., i_{r-1}$, $1 \leq i_1 < ... < i_{r-1} \leq n$, nonzero integers $a_{i_1}, ..., a_{i_{r-1}} \in \mathbb{Z}_m$ such that for every word $v \in C_n$ of weight $t+r-1$ holds

$$v(i_1) = a_{i_1}, ..., v(i_{r-1}) = a_{i_{r-1}}$$

except possibly for one position $i_j \in \{i_1, ..., i_{r-1}\}$, where we may have a nonzero entry $v(i_j) \neq a_{i_j}$.
Consider the following cases.

Case 1: There is $l \in \{i_1, ..., i_{r-1}\}$ such that for every word $v \in C_n$, $w(v) = t+r-1$, $v(j) = a_{i_j}$ for every $j \in \{i_1, ..., i_{r-1}\} \setminus \{l\}$.

We have

$$B_3 \leq \binom{n-(r-1)}{t} (m-1)^{t+1} \leq \frac{n^t}{t!} (m-1)^{t+1}, \tag{4.9}$$

which by (4.8) leads to the same contradiction as above for $r = 1$. Observe that this case also includes the situation, where all words in C_n of weight

$t+r-1$ have the same entries in positions $i_1, ..., i_{r-1}$. So it remains to consider the case, where we find two words $v, v' \in C_n$ of weight $t+r-1$ with different exceptional positions in $\{i_1, ..., i_{r-1}\}$.

Case 2: There are $v, v' \in C_n,\ w(v) = w(v') = t+r-1$, such that $v(l) \neq a_{i_l}$, $v'(l') \neq a_{i_{l'}}$ for some $l \neq l'$, $l, l' \in \{i_1, ..., i_{r-1}\}$.

Let $B = \{i_1, ..., i_{r-1}\}$, $B' = \{1, ..., n\} \setminus B$ and

$$p = |supp(v|B') \cap supp(v'|B')|.$$

We have $d(v|B, v'|B) = 2$. Now $v, v' \in C_n$, $d(v, v') \leq 2t+1$, implies $p \geq 1$. Suppose $z \in C_n$, $w(z) = t+r-1$ and z has an exceptional position $j \in B$, $z(j) \neq a_{i_j}$. We claim that z must have a nonzero entry in

$$A = supp(v|B') \cup supp(v'|B') = \{i'_1, ..., i'_{2t-p}\}.$$

W.l.o.g. we may assume $j \neq l$. Then $d(z|B, v|B) = 2$ and the supports of z and v must meet in B' because of $d(z, v) \leq 2t+1$. So z has a nonzero entry in A.
Let B'_3 be the number of all words $z \in C_n$, $w(z) = t+r-1$, with an exceptional position $j \in B$. We have $|A| = 2t-p \leq 2t$, therefore

$$\begin{aligned}
B'_3 &\leq (r-1)(m-1)\sum_{j'=1}^{t}\binom{|A|}{j'}(m-1)^{j'}\binom{n-(r-1)-|A|}{t-j'}(m-1)^{t-j'} \\
&\leq (r-1)(m-1)^{t+1}\sum_{j'=1}^{t}|A|^{j'}(n-(r-1)-|A|)^{t-j'} \\
&\leq (r-1)(m-1)^{t+1}\sum_{j'=1}^{t}(2t)^t n^{t-1} \\
&\leq (r-1)(m-1)^{t+1}t(2t)^t n^{t-1}.
\end{aligned}$$

Now $1 \leq r \leq t+1$ implies

$$B'_3 \leq (m-1)^{t+1}2^t t^{t+2} n^{t-1}.$$

Thus for $n \geq n'_1 = t!(2t)^{t+2}m^2 + n'_0$ we have

$$B'_3 \leq \frac{1}{4m^2}(m-1)^{t+1}\frac{n^t}{t!}. \tag{4.10}$$

Let B_3'' be the number of words $y \in C_n$, $w(y) = t + r - 1$, without an exceptional position in B, i.e.

$$y(i_j) = a_{i_j} \text{ for every } j = 1, ..., r-1.$$

We have

$$B_3'' \leq \binom{n-(r-1)}{t}(m-1)^t \leq (m-1)^{t+1}\frac{n^t}{t!}.$$

For the number $B_3 = B_3' + B_3''$ of all words of weight $t+r-1$ in C_n we obtain

$$B_3 \leq (1 + \frac{1}{4m^2})(m-1)^{t+1}\frac{n^t}{t!}.$$

Then by (4.8) we have for $n \geq n_1'$

$$\frac{2(m^2+m-1)-1}{2m^2 t!}(m-1)^{t+1} \leq B_3 \leq (1+\frac{1}{4m^2})\frac{(m-1)^{t+1}}{t!},$$

$$2(m^2+m-1) - 1 \leq 2m^2(1 + \frac{1}{4m^2}),\ 2m - 3 \leq \frac{1}{2},$$

which is a contradiction for $m \geq 2$.
So we know that M is a finite set. This means that there is $n_2 \geq n_1$ such that for $n \geq n_2$ every maximal clique $C \subseteq HG(m, n, 2t+1)$, $\overline{0} \in C_n$, has the property:
The number of words of maximum weight $t + r$ in C , $1 \leq r \leq t+1$, is at least $\frac{1}{2(m^2+m-1)} c_2 n^t$.

We apply again Lemma 4.1 with $s = r$. Then there is $n_0'' \geq n_2$ such that for $n \geq n_0''$ there are positions $i_1, ..., i_r$, $1 \leq i_1 < ... < i_r \leq n$, nonzero integers $a_{i_1}, ..., a_{i_r} \in \mathbb{Z}_m$ and $2t+2$ words $u^{(1)}, ..., u^{(2t+2)}$ of weight $t+r$ in C_n such that

$$u^{(j)}(i_1) = a_{i_1}, ..., u^{(j)}(i_r) = a_{i_r}$$

for every j, $1 \leq j \leq 2t+2$ and the supports of these words intersect exactly in positions $i_1, ..., i_r$. W.l.o.g. suppose $i_1 = 1, ..., i_r = r$ and $a_{i_1} = ... = a_{i_r} = 1$. Also for every word $v \in C_n$ of weight $t + r$ holds

$$v(1) = 1, ..., v(r) = 1$$

except possibly for one position $l \in \{1, ..., r\}$, where we may have a nonzero entry $v(l) \neq 1$.
We assume now $r > 1$. Let B_4 be the number of words of weight $t+r$ in C_n. Similarly as above for the words of weight $t+r-1$ we consider the following

cases.

Case 1: There is $l \in \{i_1, ..., i_r\}$ such that for every word $v \in C_n$, $w(v) = t+r$, $v(j) = a_{i_j},\ j \in \{i_1, ..., i_r\} \setminus \{l\}$.

We have

$$B_4 \leq \binom{n-r}{t}(m-1)^{t+1} \leq \frac{n^t}{t!}(m-1)^{t+1}. \tag{4.11}$$

This case also includes the situation, where all words of weight $t+r$ in C_n have the same entries in positions $i_1, ..., i_r$. So it remains to consider the case, where we find two words $v, v' \in C_n$ of weight $t+r$ with different exceptional positions in $\{i_1, ..., i_r\}$.

Case 2: There are $v, v' \in C_n,\ w(v) = w(v') = t+r$, such that $v(l) \neq a_{i_l}$, $v'(l') \neq a_{i_{l'}}$ for some $l \neq l'$, $l, l' \in \{i_1, ..., i_r\}$.

Let $B = \{i_1, ..., i_r\}$, $B' = \{1, ..., n\} \setminus B$ and

$$p = |supp(v|B') \cap supp(v'|B')|.$$

We have $d(v|B, v'|B) = 2$. Now $v, v' \in C_n$, $d(v, v') \leq 2t+1$, implies $p \geq 1$. Suppose $z \in C_n$, $w(z) = t+r$ and z has an exceptional position $j \in B$ such that $z(j) \neq a_{i_j}$. We claim that z must have a nonzero entry in

$$A = supp(v|B') \cup supp(v'|B') = \{i'_1, ..., i'_{2t-p}\}.$$

W.l.o.g. we may assume $j \neq l$. Then $d(z|B, v|B) = 2$ and the supports of z and v must meet in B' because of $d(z, v) \leq 2t+1$. So z has a nonzero entry in A.
Let B'_4 be the number of all words $z \in C_n$ with an exceptional position $j \in B$.

We have $|A| = 2t - p \leq 2t$, therefore

$$\begin{aligned} B_4' &\leq r(m-1)\sum_{j'=1}^{t}\binom{|A|}{j'}(m-1)^{j'}\binom{n-r-|A|}{t-j'}(m-1)^{t-j'} \\ &\leq r(m-1)^{t+1}\sum_{j'=1}^{t}|A|^{j'}(n-r-|A|)^{t-j'} \\ &\leq r(m-1)^{t+1}\sum_{j'=1}^{t}(2t)^t n^{t-1} \\ &\leq r(m-1)^{t+1}t(2t)^t n^{t-1}. \end{aligned}$$

Now $1 \leq r \leq t+1$ implies

$$B_4' \leq (m-1)^{t+1}(t+1)t^{t+1}2^t n^{t-1}.$$

Thus for $n \geq n_2' = (t+1)!(2t)^{t+2}m^2 + n_0''$ we have

$$B_4' \leq \frac{1}{4m^2}(m-1)^{t+1}\frac{n^t}{t!}.$$

Let B_4'' be the number of words $y \in C_n$ without an exceptional position in B, i.e.

$$y(i_j) = a_{i_j} \quad \text{for every} \quad j = 1, ..., r.$$

We have

$$B_4'' \leq \binom{n-r}{t}(m-1)^t \leq (m-1)^{t+1}\frac{n^t}{t!}.$$

For the number $B_4 = B_4' + B_4''$ of all words of weight $t + r$ in C_n we obtain

$$B_4 \leq (1 + \frac{1}{4m^2})(m-1)^{t+1}\frac{n^t}{t!}.$$

According to (4.11) this upper bound is valid also in Case 1 for $n \geq n_2'$.

Consider A_{r-1}', the number of words of weight $t + r - 1$ in C_n. Then we have by (4.1) and (4.6) for $n \geq n_2'$:

$$\begin{aligned}
A'_{r-1} &= |C| - B_4 - B_1 \\
&\geq \frac{(m+1)(m-1)^{t+1}}{t!m} n^t - (1 + \frac{1}{4m^2})(m-1)^{t+1}\frac{n^t}{t!} - \frac{(m-1)^{t+1}}{m^2 t!} n^t \\
&\geq \frac{4m(m+1) - 4m^2 - 1 - 4}{4m^2 t!}(m-1)^{t+1} n^t \\
&\geq \frac{4m-5}{4m^2 t!}(m-1)^{t+1} n^t \\
&\geq \frac{m-1}{4m^2 t!}(m-1)^{t+1} n^t = \frac{(m-1)^{t+2}}{4m^2 t!} n^t.
\end{aligned}$$

We apply again Lemma 4.1 with $s = r-1$. Then Lemma 4.1 guarantees the existence of $n_0 \in \mathbb{N}$, $n_0 \geq max\{n_2', n_0''\}$, such that for $n \geq n_0$ there are positions $k_1, ..., k_{r-1}$, $1 \leq k_1 < ... < k_{r-1} \leq n$, nonzero integers $a'_{k_1}, ..., a'_{k_{r-1}} \in \mathbb{Z}_m$ and $2t+2$ words $v^{(1)}, ..., v^{(2t+2)}$ of weight $t+r-1$ in C_n such that

$$v^{(j)}(k_1) = a'_{k_1}, ..., v^{(j)}(k_{r-1}) = a'_{k_{r-1}}$$

for every j, $1 \leq j \leq 2t+2$ and the supports of these words intersect exactly in positions $k_1, ..., k_{r-1}$.

According to Lemma 3.2(1), for every word $v^{(j)} \in C$, $1 \leq j \leq 2t+2$, we have

$$|overlap(v^{(j)}, u^{(i)})| \geq \lceil \frac{r+r-1-1}{2} \rceil = r-1$$

for every i, $1 \leq i \leq 2t+2$. If $|supp(v^{(j)}) \cap \{1, ..., r\}| < r-1$ then $supp(v^{(j)})$ must have an additional common position with the support of every word $u^{(i)}$, $1 \leq i \leq 2t+2$. But this would raise the weight of $v^{(j)}$ at least up to $2t+2 > t+r-1$, which is a contradiction.

Therefore for every j, $1 \leq j \leq 2t+2$, we have

$$|supp(v^{(j)}) \cap \{1, ..., r\}| \geq r-1.$$

Every $v^{(j)}$, $1 \leq j \leq 2t+2$, has at most one exceptional position in $\{1, ..., r\}$, in which it may have the entry zero. As $2t+2 > r$, there are two words $v^{(i)}, v^{(j)}$, $i \neq j$, with the same exceptional position or such that one of these words has no exceptional position. But this means

$$|supp(v^{(i)}) \cap supp(v^{(j)}) \cap \{1, ..., r\}| = r-1.$$

By

$$supp(v^{(i)}) \cap supp(v^{(j)}) = \{k_1, ..., k_{r-1}\}$$

we conclude

$$\{k_1, ..., k_{r-1}\} \subset \{1, ..., r\}.$$

W.l.o.g. suppose that $k_1 = 1, ..., k_{r-1} = r - 1$. We show that

$$v^{(j)}(1) = 1, ..., v^{(j)}(r-1) = 1$$

for every $j,\ 1 \le j \le 2t+2$.
Suppose that there is $l \in \{1, ..., r-1\}$ such that $v^{(j)}(l) \ne 1$ for every $j,\ 1 \le j \le 2t+2$. We know the supports of these words are disjoint in positions $r, ..., n$. Then there exists $j,\ 1 \le j \le 2t+2$, such that $v^{(j)}(r) = 0$. Let $B = \{1, ..., r\}$ and $B' = \{1, ..., n\} \setminus B$. For every i, $1 \le i \le 2t+2$, we have

$$d(v^{(j)}|B, u^{(i)}|B) = 2.$$

As $w(v^{(j)}) < 2t+2$, there exists i, $1 \le i \le 2t+2$, such that

$$supp(v^{(j)}|B') \cap supp(u^{(i)}|B') = \phi.$$

Then we have

$$d(v^{(j)}|B', u^{(i)}|B') = 2t$$

thus

$$d(v^{(j)}, u^{(i)}) = d(v^{(j)}|B, u^{(i)}|B) + d(v^{(j)}|B', u^{(i)}|B') = 2t+2.$$

But $v^{(j)}$ and $u^{(i)}$ are in C therefore $d(v^{(j)}, u^{(i)}) = 2t+1$ and it is a contradiction. □

Corollary 4.1. *The statement of Theorem 4.1 after the asumptions can be supplemented by the following facts.*

1. *Let $v \in C$ be a word of weight $t+s,\ s = -r+2j,\ 1 \le j \le r$. Then we have*

$$|supp(v) \cap \{i_1, ..., i_r\}| \ge \frac{r+s}{2} = j.$$

Suppose that

$$|supp(v) \cap \{i_1, ..., i_r\}| = \frac{r+s}{2} + k, \quad 0 \le k \le r - \frac{r+s}{2}.$$

Denote by X the number of positions $i_l \in \{i_1, ..., i_r\} \cap supp(v)$ with $v(i_l) \ne a_{i_l}$. Then we have

$$X \le 2k+1.$$

2. *Let $r > 1$ and $v \in C$ be a word of weight $t+s,\ s = -r+2j+1,\ 1 \le j \le r-1$. Then we have*

$$|supp(v) \cap \{k_1, ..., k_{r-1}\}| \ge \frac{r+s-1}{2} = j.$$

Suppose that

$$|supp(v) \cap \{k_1, ..., k_{r-1}\}| = \frac{r+s-1}{2} + k, \quad 0 \le k \le r - 1 - \frac{r+s-1}{2}.$$

Denote by X' *the number of positions* $i_{l'} \in \{k_1, ..., k_{r-1}\} \cap supp(v)$ *with* $v(i_{l'}) \neq a_{i_{l'}}$. *Then we have*

$$X' \le 2k + 1.$$

Proof. Let n_0, positions $i_1, ..., i_r, k_1, ..., k_{r-1}$ and words

$$u^{(1)}, ..., u^{(2t+2)}, v^{(1)}, ..., v^{(2t+2)}$$

be determined as in Theorem 4.1. W.l.o.g. we may assume $i_1 = 1, ..., i_r = r, k_1 = 1, ..., k_{r-1} = r-1$. Words $u^{(1)}, ..., u^{(2t+2)}$ have weight $t+r$ and the supports of these words intersect exactly in positions 1,...,r. Also words $v^{(1)}, ..., v^{(2t+2)}$ have weight $t+r-1$ and the supports of these words intersect exactly in positions 1,...,$r-1$.

1. Let $v \in C$, $w(v) = t+s$, $s = -r+2j$ $(1 \le j \le r)$. By Lemma 3.2(1), we have

$$|overlap(u^{(i)}, v)| \ge \lceil \frac{r+s-1}{2} \rceil = j = \frac{r+s}{2}$$

for every i, $1 \le i \le 2t+2$. If $|supp(v) \cap \{1, ..., r\}| < j$ then $supp(v)$ must have an additional common position with the support of every word $u^{(i)}$, $1 \le i \le 2t+2$. But this would raise the weight of v at least up to $2t+2$, which is a contradiction.

Now suppose that

$$|supp(v) \cap \{1, ..., r\}| = \frac{r+s}{2} + k, \quad 0 \le k \le r - \frac{r+s}{2}.$$

Consider $B_1 = \{1, ..., r\}$. For every i, $1 \le i \le 2t+2$, we have

$$d(v|B_1, u^{(i)}|B_1) = r - \frac{r+s}{2} - k + X.$$

Let $B_1' = \{1, ..., n\} \setminus B_1$. As $w(v) < 2t+2$, there exists i, $1 \le i \le 2t+2$, such that

$$supp(v|B_1') \cap supp(u^{(i)}|B_1') = \phi.$$

Then we have

$$d(v|B_1', u^{(i)}|B_1') = t + t + s - \frac{r+s}{2} - k,$$

thus

$d(v, u^{(i)}) = d(v|B_1, u^{(i)}|B_1) + d(v|B_1', u^{(i)}|B_1') = 2t + r + s - (r + s) - 2k + X.$

But v and $u^{(i)}$ are in C, therefore

$$d(v, u^{(i)}) = 2t - 2k + X \leq 2t + 1$$

and

$$X \leq 2k + 1.$$

2. Let $v \in C$, $w(v) = t + s$, $s = -r + 2j + 1$ $(1 \leq j \leq r - 1)$. By Lemma 3.2(1), we have

$$|overlap(v^{(i)}, v)| \geq \lceil \frac{r + s - 2}{2} \rceil = j = \frac{r + s - 1}{2}$$

for every i, $1 \leq i \leq 2t + 2$. If $|supp(v) \cap \{1, ..., r - 1\}| < j$ then similarly to part 1 we get a contradiction.

Now suppose that

$$|supp(v) \cap \{1, ..., r - 1\}| = \frac{r + s - 1}{2} + k, \quad 0 \leq k \leq r - 1 - \frac{r + s - 1}{2}.$$

Consider $B_2 = \{1, ..., r - 1\}$.For every i, $1 \leq i \leq 2t + 2$, we have

$$d(v|B_2, v^{(i)}|B_2) = r - 1 - \frac{r + s - 1}{2} - k + X'.$$

Let $B_2' = \{1, ..., n\} \setminus B_2$. As $w(v) < 2t + 2$, there exists i, $1 \leq i \leq 2t + 2$, such that

$$supp(v|B_2') \cap supp(v^{(i)}|B_2') = \phi.$$

Then we have

$$d(v|B_2', v^{(i)}|B_2') = t + t + s - \frac{r + s - 1}{2} - k,$$

thus

$d(v, v^{(i)}) = d(v|B_2, v^{(i)}|B_2) + d(v|B_2', v^{(i)}|B_2') = 2t + r + s - (r + s - 1) - 2k - 1 + X'.$

But v and $v^{(i)}$ are in C, therefore

$$d(v, v^{(i)}) = 2t - 2k + X' \leq 2t + 1$$

and

$$X' \leq 2k + 1.$$

$\square$

Theorem 4.2. *For integers m, t, $m \geq 2$, $t \in \{1, 2\}$, there is a positive integer n_0' (depending only on t and m) such that for every integer $n \geq n_0'$ the following statement is true:*
Suppose C is a maximal clique of $HG(m, n, 2t + 1)$, $\overline{0} \in C$ and $t + r$, $1 \leq r \leq t + 1$, is the maximum weight of a word in C. Then

$$\binom{n-r}{t}(m-1)^{t+1}$$

is an upper bound for the number of words of weight $t + r$ in C.
This statement is true for every $t \geq 1$, if $m = 2$, also for every $t \geq 1$, $m \geq 2$, if $r = 1$.

Proof. Let n_0 be choosen according to Theorem 4.1 and $n \geq n_0$. By Theorem 4.1 there are positions $i_1, ..., i_r$ and nonzero integers $a_{i_1}, ..., a_{i_r}$ and $2t + 2$ words $u^{(1)}, ..., u^{(2t+2)}$ of weight $t + r$, w.l.o.g. $i_1 = 1, ..., i_r = r$ and $a_{i_j} = 1$ for $1 \leq j \leq r$, such that

$$u^{(j)}(1) = 1, ..., u^{(j)}(r) = 1$$

for every j, $1 \leq j \leq 2t + 2$ and the supports of these words intersect exactly in positions 1,...,r.
We denote by A the number of words of weight $t + r$ in C.
Let $v \in C$ be a word of weight $t + r$. By Corollary 4.1(1) we have

$$\{1, ..., r\} \subseteq supp(v).$$

If $m = 2$ then for every word v of weight $t + r$ in C we have

$$v(1) = 1, ..., v(r) = 1.$$

Thus for every $n \geq n_0' = n_0$ we conclude

$$A \leq \binom{n-r}{t}.$$

Therefore we consider $m \geq 3$.
In the notation of Corollary 4.1(1), we have $s = r$, $k = 0$ and $X \leq 1$. This means for every word $v \in C$ of weight $t + r$ holds

$$v(1) = 1, ..., v(r) = 1$$

except possibly for one position $j \in \{1, ..., r\}$, where we may have a nonzero entry $v(j) \neq 1$.

If $r = 1$ then

$$A \leq \binom{n-1}{t}(m-1)^{t+1}$$

and we choose $n_0' = n_0$.
Therefore we suppose $2 \leq r \leq t+1 \leq 3$. Now we consider the following cases.

Case 1: Every $v \in C$ of weight $t+r$ has $v(1) = ... = v(r) = 1$.

We have

$$A \leq \binom{n-r}{t}(m-1)^{t}$$

and for every $n \geq n_0$ we conclude

$$A \leq \binom{n-r}{t}(m-1)^{t} \leq \binom{n-r}{t}(m-1)^{t+1}.$$

Case 2: There is $j \in \{1, ..., r\}$ such that for every word $v \in C$, $w(v) = t+r$, $v(i) = 1$ for every $i \neq j$, $v(j) \neq 0$.

We conclude for $n \geq n_0$

$$A \leq \binom{n-r}{t}(m-1)^{t+1}.$$

Case 3: There are $v, v' \in C,\ w(v) = w(v') = t+r$, such that $v(l) \neq 0, 1$, $v'(l') \neq 0, 1$ for some $l \neq l'$, $l, l' \in \{1, ..., r\}$.

Let $B = \{1, ..., r\}$ and $B' = \{1, ..., n\} \setminus B$. We have $d(v|B, v'|B) = 2$. But $v, v' \in C$ and $d(v, v') \leq 2t+1$, therefore

$$p = |supp(v|B') \cap supp(v'|B')| \geq 1.$$

Also $|supp(v|B')| = |supp(v'|B')| = t$ implies $1 \leq p \leq t$.

Case 3.1: $t = 1$

We know $2 \leq r \leq t+1$ therefore $r = 2$. We have $p = 1$, w.l.o.g. suppose that

$$v = (a, 1, v_3, 0, ..., 0); a \neq 0, 1; v_3 \neq 0$$

and

$$v' = (1, b, v_3', 0, ..., 0); b \neq 0, 1; v_3' \neq 0.$$

Suppose that $z \in C$ is a word of weight 3, which has not both entries $z(1) = z(2) = 1$. Consider $z(2) = c \neq 0, 1$. Then we have $d(v|B, z|B) = 2$ and

$$|supp(v|B') \cap supp(z|B')| \geq 1$$

and $z(3) \neq 0$. Therefore the number of these words in C is at most

$$(m-2)(m-1).$$

Similarly, if $z(1) = c \neq 0, 1$ then we have the same result. Therefore the number of words z of weight 3 in C which have not both entries $z(1) = z(2) = 1$, is at most

$$2(m-2)(m-1). \tag{4.12}$$

The other words of weight 3 in C have both first entries equal to 1 and the number of these words in C is at most

$$(n-2)(m-1). \tag{4.13}$$

Therefore we have by (4.12),(4.13)

$$\begin{aligned} A &\leq 2(m-2)(m-1) + (n-2)(m-1) \\ &\leq 2(m-1)^2 + (n-4)(m-1). \end{aligned}$$

But for $n \geq 4$ we have $n - 4 \geq 0$ and

$$A \leq 2(m-1)^2 + (n-4)(m-1)^2 = (n-2)(m-1)^2 = \binom{n-2}{1}(m-1)^2.$$

We choose $n_0' = max\{n_0, 4\}$.

Case 3.2: $t = 2$

We have $1 \leq p \leq 2$ and $2 \leq r \leq 3$.

Case 3.2.1: $p = 1$

At first we consider $r = 2$. W.l.o.g. suppose that

$$v = (a, 1, v_3, v_4, 0, ..., 0); a \neq 0, 1; v_3, v_4 \neq 0$$

and

$$v' = (1, b, v_3', 0, v_5', 0, ..., 0); b \neq 0, 1; v_3', v_5' \neq 0.$$

Suppose that $z \in C$ is a word of weight 4, which has not both entries $z(1) = z(2) = 1$. Consider $z(2) = c \neq 0, 1$. Then we have $d(v|B, z|B) = 2$ and

$$|supp(v|B') \cap supp(z|B')| \geq 1.$$

We conclude at least $z(3) \neq 0$ or $z(4) \neq 0$. Therefore the number of these words is at most

$$(m-2)(m-1)(n-3)(m-1) + (m-2)(m-1)(n-4)(m-1)$$

$$=$$

$$(m-2)(m-1)^2(2n-7).$$

Similarly, if $z(1) = c \neq 0, 1$ we have the same result. Therefore the number of words z of weight 4 in C which have not both entries $z(1) = z(2) = 1$, is at most

$$2(m-2)(m-1)^2(2n-7). \tag{4.14}$$

The other words of weight 4 in C have both first entries equal to 1 and the number of these words in C is at most

$$\binom{n-2}{2}(m-1)^2. \tag{4.15}$$

We have by (4.14),(4.15)

$$\begin{aligned}
A &\leq 2(m-2)(m-1)^2(2n-7) + \binom{n-2}{2}(m-1)^2 \\
&\leq 2(2n-7)(m-1)^3 - 2(2n-7)(m-1)^2 + \binom{n-2}{2}(m-1)^2 \\
&\leq 2(2n-7)(m-1)^3 + \frac{(n-2)(n-3) - 4(2n-7)}{2}(m-1)^2.
\end{aligned}$$

For $n \geq 10$ we have $(n-2)(n-3) > 4(2n-7)$, thus

$$\begin{aligned}
A &\leq 2(2n-7)(m-1)^3 + \frac{(n-2)(n-3) - 4(2n-7)}{2}(m-1)^3 \\
&\leq \frac{(n-2)(n-3)}{2}(m-1)^3 = \binom{n-2}{2}(m-1)^3.
\end{aligned}$$

We choose $n_0' = max\{n_0, 10\}$.

Now we consider $r = 3$. W.l.o.g. suppose that

$$v = (a, 1, 1, v_4, v_5, 0, ..., 0); a \neq 0, 1; v_4, v_5 \neq 0$$

and

$$v' = (1, b, 1, v_4', 0, v_6', 0, ..., 0); b \neq 0, 1; v_4', v_6' \neq 0.$$

Suppose that $z \in C$ is a word of weight 5, which has not all entries $z(1) = z(2) = z(3) = 1$. Consider $z(2) = c \neq 0, 1$. Then we have $d(v|B, z|B) = 2$ and

$$|supp(v|B') \cap supp(z|B')| \geq 1.$$

We conclude at least $z(4) \neq 0$ or $z(5) \neq 0$. Therefore the number of these words is at most

$$(m-2)(m-1)(n-4)(m-1) + (m-2)(m-1)(n-5)(m-1)$$

$$=$$

$$(m-2)(m-1)^2(2n-9).$$

Similarly, if $z(1) = c \neq 0, 1$ we have the same result.
Consider $z(3) = c \neq 0, 1$. Then we have

$$d(v|B, z|B) = 2, \; d(v'|B, z|B) = 2,$$

$$|supp(v|B') \cap supp(z|B')| \geq 1, \; |supp(v'|B') \cap supp(z|B')| \geq 1.$$

We conclude that $z(4) \neq 0$ or $z(4) = 0$ and both $z(5)$ and $z(6)$ are not equal to zero. Therefore the number of these words is at most

$$(m-2)(m-1)(n-4)(m-1) + (m-2)(m-1)(m-1)$$

$$=$$

$$(m-2)(m-1)^2(n-3).$$

The number of words z of weight 5 in C which have not all entries $z(1) = z(2) = z(3) = 1$, is at most

$$2(m-2)(m-1)^2(2n-9) + (m-2)(m-1)^2(n-3)$$

$$=$$

$$(m-2)(m-1)^2(5n-21). \tag{4.16}$$

The other words of weight 5 in C have all entries equal to 1 in positions $1, 2, 3$ and the number of these words in C is at most

$$\binom{n-3}{2}(m-1)^2. \tag{4.17}$$

We have by (4.16),(4.17)

$$\begin{aligned} A &\leq (m-2)(m-1)^2(5n-21) + \binom{n-3}{2}(m-1)^2 \\ &\leq (5n-21)(m-1)^3 - (5n-21)(m-1)^2 + \binom{n-3}{2}(m-1)^2 \\ &\leq (5n-21)(m-1)^3 + \frac{(n-3)(n-4)-2(5n-21)}{2}(m-1)^2. \end{aligned}$$

But for $n \geq 13$ we have $(n-3)(n-4) > 2(5n-21)$, thus

$$\begin{aligned} A &\leq (5n-21)(m-1)^3 + \frac{(n-3)(n-4)-2(5n-21)}{2}(m-1)^3 \\ &\leq \frac{(n-3)(n-4)}{2}(m-1)^3 = \binom{n-3}{2}(m-1)^3. \end{aligned}$$

We choose $n'_0 = max\{n_0, 13\}$.

Case 3.2.2: $p = 2$

At first we consider $r = 2$. W.l.o.g. suppose that

$$v = (a, 1, v_3, v_4, 0, ..., 0); a \neq 0, 1; v_3, v_4 \neq 0$$

and

$$v' = (1, b, v'_3, v'_4, 0, ..., 0); b \neq 0, 1; v'_3, v'_4 \neq 0.$$

Suppose that $z \in C$ has weight 4 and not both entries $z(1) = z(2) = 1$. Consider $z(2) = c \neq 0, 1$. Then we have $d(v|B, z|B) = 2$ and

$$|supp(v|B') \cap supp(z|B')| \geq 1.$$

If $|supp(v|B') \cap supp(z|B')| = 1$ then this case is equivalent to Case 3.2.1 with $r = 2$. Therefore suppose

$$|supp(v|B') \cap supp(z|B')| = 2.$$

The number of these words in C is at most

$$(m-2)(m-1)^2.$$

Similarly, if $z(1) = c \neq 0, 1$ we have the same result. Therefore the number of words z of weight 4 in C which have not both entries $z(1) = z(2) = 1$, is at most

$$2(m-2)(m-1)^2. \tag{4.18}$$

The other words of weight 4 in C have both first entries equal to 1 and the number of these words in C is at most

$$\binom{n-2}{2}(m-1)^2. \tag{4.19}$$

We have by (4.18),(4.19)

$$\begin{aligned} A &\leq 2(m-2)(m-1)^2 + \binom{n-2}{2}(m-1)^2 \\ &\leq 2(m-1)^3 - 2(m-1)^2 + \binom{n-2}{2}(m-1)^2 \\ &\leq 2(m-1)^3 + \frac{(n-2)(n-3)-4}{2}(m-1)^2. \end{aligned}$$

For $n \geq 5$ we have $(n-2)(n-3) > 4$, thus

$$A \leq 2(m-1)^3 + \frac{(n-2)(n-3)-4}{2}(m-1)^3 = \binom{n-2}{2}(m-1)^3.$$

We choose $n_0' = max\{n_0, 5\}$.

Now we consider $r = 3$. W.l.o.g. suppose that

$$v = (a, 1, 1, v_4, v_5, 0, ..., 0); a \neq 0, 1; v_4, v_5 \neq 0$$

and

$$v' = (1, b, 1, v_4', v_5', 0, ..., 0); b \neq 0, 1; v_4', v_5' \neq 0.$$

Suppose that $z \in C$ is a word of weight 5, which has not all entries $z(1) = z(2) = z(3) = 1$. Consider $z(2) = c \neq 0, 1$. Then we have $d(v|B, z|B) = 2$ and

$$|supp(v|B') \cap supp(z|B')| \geq 1.$$

If $|supp(v|B') \cap supp(z|B')| = 1$ then this case is equivalent to Case 3.2.1 with $r = 3$. Therefore suppose

$$|supp(v|B') \cap supp(z|B')| = 2.$$

The number of these words in C is at most

$$(m-2)(m-1)^2.$$

Similarly, if $z(1) = c \neq 0, 1$ or $z(3) = c \neq 0, 1$ we have the same result. Therefore the number of words z of weight 5 in C which have not all entries $z(1) = z(2) = z(3) = 1$, is at most

$$3(m-2)(m-1)^2. \tag{4.20}$$

The other words of weight 5 in C have entries equal to 1 in all positions $1, 2, 3$. The number of these words in C is at most

$$\binom{n-3}{2}(m-1)^2. \tag{4.21}$$

We have by (4.20),(4.21)

$$\begin{aligned} A &\leq 3(m-2)(m-1)^2 + \binom{n-3}{2}(m-1)^2 \\ &\leq 3(m-1)^3 - 3(m-1)^2 + \binom{n-3}{2}(m-1)^2 \\ &\leq 3(m-1)^3 + \frac{(n-3)(n-4)-6}{2}(m-1)^2. \end{aligned}$$

For $n \geq 6$ we have $(n-3)(n-4) \geq 6$, thus

$$A \leq 3(m-1)^3 + \frac{(n-3)(n-4)-6}{2}(m-1)^3 = \binom{n-3}{2}(m-1)^3.$$

We choose $n'_0 = max\{n_0, 6\}$. $\square$

Lemma 4.2. *For integers $m, t,\ m \geq 2,\ t \in \{1, 2\}$, there is a positive integer n''_0 (depending only on t and m) such that for every integer $n \geq n''_0$ the following statement is true:*
Suppose C is a maximal clique of $HG(m, n, 2t+1)$, $\overline{0} \in C$ and $t+r$, $1 \leq r \leq 2$, is the maximum weight of a word in C, then

$$|C| = \sum_{j=0}^{t} \binom{n}{j}(m-1)^j + \binom{n-1}{t}(m-1)^{t+1}.$$

Remark. Observe that Lemma 4.2 does not necessarily mean that for every m, t, r (as in Lemma 4.2) there is a maximal clique $C \subset HG(m, n, 2t+1)$ satisfying the above equation for $|C|$.

Proof of Lemma 4.2. Suppose that C is a maximal clique of $HG(m, n, 2t+1)$, $\overline{0} \in C$ and $t+r$ is the maximum weight of a word in C, $1 \leq r \leq 2$. We know for $n \geq 2t+1$ by Proposition 1.2

$$|C| \geq \sum_{j=0}^{t} \binom{n}{j}(m-1)^j + \binom{n-1}{t}(m-1)^{t+1} = \omega_0(m, n, 2t+1).$$

So it suffices to show

$$|C| \leq \omega_0(m, n, 2t+1).$$

By Theorem 4.1 there is $n_0 \in \mathbb{N}$ with the following property. If $n \geq n_0$ there are positions $i_1, ..., i_r, k_1, ..., k_{r-1}$ and nonzero integers $a_{i_1}, ..., a_{i_r}$, w.l.o.g. $i_1 = 1, ..., i_r = r, k_1 = 1, ..., k_{r-1} = r-1$ and $a_{i_j} = 1$ for $1 \leq j \leq r$, and $2t+2$ words $u^{(1)}, ..., u^{(2t+2)}$ of weight $t+r$ and $2t+2$ words $v^{(1)}, ..., v^{(2t+2)}$ of weight $t+r-1$ such that

$$u^{(j)}(1) = 1, ..., u^{(j)}(r) = 1$$

for every j, $1 \leq j \leq 2t+2$ and

$$v^{(j)}(1) = 1, ..., v^{(j)}(r-1) = 1$$

for every j, $1 \leq j \leq 2t+2$. Also the supports of $u^{(1)}, ..., u^{(2t+2)}$ intersect exactly in positions $1, ..., r$ and the supports of $v^{(1)}, ..., v^{(2t+2)}$ intersect exactly in positions $1, ..., r-1$.
Let n_0' be determined by Theorem 4.2. We choose $n_0'' = n_0' \geq n_0$ and suppose $n \geq n_0''$.
Let A_1 be the number of all words of weight at most $t-r+1$. These words have at most distance $2t+1$ to every word in C. They must be contained in C, because C is maximal. We have

$$A_1 = \sum_{j=0}^{t-r+1} \binom{n}{j}(m-1)^j. \tag{4.22}$$

If we denote by A_2 the number of words in C of weight at least $t-r+2$ then we have

$$|C| = A_1 + A_2.$$

If $r = 1$, then A_2 is the number of words of weight $t+r$ in C, which by Theorem 4.2 satisfies

$$A_2 \leq \binom{n-1}{t}(m-1)^{t+1}$$

for every $t \geq 1$, $m \geq 2$. So we conclude by (4.22)

$$|C| = A_1 + A_2 \leq \sum_{j=0}^{t} \binom{n}{j}(m-1)^j + \binom{n-1}{t}(m-1)^{t+1}.$$

Therefore we consider $r = 2$.
Let A'_s be the number of all words in C of weight $t+s,\ 0 \leq s \leq 2$. Then we have

$$A_2 = \sum_{s=0}^{2} A'_s. \tag{4.23}$$

Consider the following cases.

Case 1: $t = 1$
We have by (4.22)

$$A_1 = \sum_{j=0}^{0} \binom{n}{j}(m-1)^j = 1.$$

Also by Theorem 4.2 we have

$$A'_2 \leq \binom{n-2}{1}(m-1)^2.$$

For $s = 1$ by Corollary 4.1(2), every word v of weight $t+s = 2$, must have

$$|supp(v) \cap \{1\}| \geq \frac{r+s-1}{2} = \frac{2+1-1}{2} = 1,$$

also $k = 0$ and $X' \leq 1$.
The number of words v of weight 2 with $X' = 0$ in C is at most

$$\binom{n-1}{1}(m-1). \tag{4.24}$$

Let v be a word of weight 2 in C with $X' = 1$. Let $B = \{1, 2\}$ and $B' = \{1, ..., n\} \setminus B$. Suppose $v(2) = 0$. Then we have $d(v|B, u^{(i)}|B) = 2$ for every i, $1 \leq i \leq 4$. As $w(v) < 4$ there exists i, $1 \leq i \leq 4$, such that

$$supp(v|B') \cap supp(u^{(i)}|B') = \phi.$$

Therefore $d(v|B', u^{(i)}|B') = 2$ and we coclude

$$d(v, u^{(i)}) = d(v|B, u^{(i)}|B) + d(v|B', u^{(i)}|B') = 2 + 2 = 4,$$

which is a contradiction.
Therefore $v(2) \neq 0$ and the number of these words in C is at most

$$(m-2)(m-1). \tag{4.25}$$

Thus we have by (4.24), (4.25)

$$A'_1 \leq \binom{n-1}{1}(m-1) + (m-2)(m-1).$$

For $s = 0$ by Corollary 4.1(1), every word v of weight $t + s = 1$, must have

$$|supp(v) \cap \{1, 2\}| \geq \frac{r+s}{2} = \frac{2+0}{2} = 1,$$

also $k = 0$ and $X \leq 1$, which implies

$$A'_0 \leq 2(m-1).$$

Therefore we have by (4.23)

$$\begin{aligned}
A_2 &\leq \binom{n-2}{1}(m-1)^2 + \binom{n-1}{1}(m-1) + (m-2)(m-1) + 2(m-1) \\
&\leq \binom{n-2}{1}(m-1)^2 + \binom{n-1}{1}(m-1) + (m-1)^2 - (m-1) + 2(m-1) \\
&\leq \binom{n-1}{1}(m-1)^2 + \binom{n}{1}(m-1).
\end{aligned}$$

Thus

$$|C| = A_1 + A_2 \leq \sum_{j=0}^{1} \binom{n}{j}(m-1)^j + \binom{n-1}{1}(m-1)^2.$$

Case 2: $t = 2$
By (4.22) We have

$$A_1 = \sum_{j=0}^{1} \binom{n}{j}(m-1)^j.$$

To estimate A_2 we establish upper bounds for A'_0, A'_1, A'_2. By Theorem 4.2 we have

$$A'_2 \leq \binom{n-2}{2}(m-1)^3.$$

Let $s = 1$ and $v \in C$ be a word of weight $w(v) = t + s = 3$. Corollary 4.1(2) implies

$$|supp(v) \cap \{1\}| \geq \frac{r+s-1}{2} = \frac{2+1-1}{2} = 1,$$

also $k = 0$ and $X' \leq 1$.
The number of words v of weight 3 with $X' = 0$ in C is at most

$$\binom{n-1}{2}(m-1)^2. \tag{4.26}$$

Let v be a word of weight 3 in C with $X' = 1$. Let $B = \{1, 2\}$ and $B' = \{1, ..., n\} \setminus B$. Suppose $v(2) = 0$. Then we have $d(v|B, u^{(i)}|B) = 2$ for every i, $1 \leq i \leq 6$. As $w(v) < 6$ there exists i, $1 \leq i \leq 6$, such that

$$supp(v|B') \cap supp(u^{(i)}|B') = \phi.$$

Therefore $d(v|B', u^{(i)}|B') = 4$ and we conclude

$$d(v, u^{(i)}) = d(v|B, u^{(i)}|B) + d(v|B', u^{(i)}|B') = 2 + 4 = 6,$$

which is a contradiction.
Therefore $v(2) \neq 0$ and the number of these words in C is at most

$$(m-2)(m-1)\binom{n-2}{1}(m-1). \tag{4.27}$$

Thus we have by (4.26), (4.27)

$$A'_1 \leq \binom{n-1}{2}(m-1)^2 + (m-2)(m-1)^2\binom{n-2}{1}.$$

For $s = 0$ by Corollary 4.1(1), every word $v \in C$ of weight $t + s = 2$, must have

$$|supp(v) \cap \{1, 2\}| \geq \frac{r+s}{2} = \frac{2+0}{2} = 1.$$

The number of words v of weight 2 with $|supp(v) \cap \{1, 2\}| = 1$ in C is at most

$$\binom{2}{1}\binom{n-2}{1}(m-1)^2.$$

Also the number of words v of weight 2 with $|supp(v) \cap \{1, 2\}| = 2$ in C is at most

$$(m-1)^2.$$

Thus we have

$$A'_0 \leq \binom{2}{1}\binom{n-2}{1}(m-1)^2 + (m-1)^2$$

and by (4.23)

$$\begin{aligned}
A_2 &\le \binom{n-2}{2}(m-1)^3 + \binom{n-1}{2}(m-1)^2 + (m-2)(m-1)^2\binom{n-2}{1} \\
&\quad + \binom{2}{1}\binom{n-2}{1}(m-1)^2 + (m-1)^2 \\
&\le \binom{n-2}{2}(m-1)^3 + \binom{n-1}{2}(m-1)^2 + \binom{n-2}{1}(m-1)^3 \\
&\quad - \binom{n-2}{1}(m-1)^2 + \binom{2}{1}\binom{n-2}{1}(m-1)^2 + (m-1)^2,
\end{aligned}$$

$$A_2 \le \left[\binom{n-2}{2} + \binom{n-2}{1}\right](m-1)^3 + \left[\binom{n-1}{2} + \binom{n-2}{1} + 1\right](m-1)^2.$$

Proposition 2.1(2) implies

$$A_2 \le \binom{n-1}{2}(m-1)^3 + \binom{n}{2}(m-1)^2$$

and we have

$$|C| = A_1 + A_2 \le \sum_{j=0}^{2}\binom{n}{j}(m-1)^j + \binom{n-1}{2}(m-1)^3.$$

□

Corollary 4.2. *For every integer m, $m \ge 2$, there is $n_0 \in \mathbb{N}$ (depending only on m) such that for every $n \ge n_0$*

$$\omega(HG(m,n,3)) = \omega_0(m,n,3) = \sum_{j=0}^{1}\binom{n}{j}(m-1)^j + \binom{n-1}{1}(m-1)^2.$$

Proof. Choose n_0'' according to Lemma 4.2 and let $n \ge n_0 = n_0''$. Suppose C is a maximal clique in $HG(m,n,3)$. By Proposition 1.1 we may assume $\overline{0} \in C$. If $1+r$ is the maximal weight of a word in C, then we have

$$2 \le 1+r \le 3,\ 1 \le r \le 2.$$

Lemma 4.2 with $t = 1$ implies

$$\omega(HG(m,n,3)) = |C| = \omega_0(m,n,3) = \sum_{j=0}^{1}\binom{n}{j}(m-1)^j + \binom{n-1}{1}(m-1)^2.$$

□

Proposition 4.1. *For every integer m, $m \geq 2$, there is $n_0 \in \mathbb{N}$ (depending only on m) such that for every $n \geq n_0$*

$$\omega(HG(m,n,5)) = \omega_0(m,n,5) = \sum_{j=0}^{2} \binom{n}{j}(m-1)^j + \binom{n-1}{2}(m-1)^3.$$

Proof. We know by Proposition 1.2 for $n \geq 5$

$$\omega(HG(m,n,5)) \geq \sum_{j=0}^{2} \binom{n}{j}(m-1)^j + \binom{n-1}{2}(m-1)^3.$$

We choose the positive integer n_0 such that for $n \geq n_0$ all statements of Theorem 4.1, Corollary 4.1, Theorem 4.2 and Lemma 4.2 hold.
Let C be a maximal clique of $HG(m,n,5)$. By Proposition 1.1 we may assume $\overline{0} \in C$. Therefore for every word $u \in C$, we have $w(u) \leq 5$. We show that

$$|C| \leq \sum_{j=0}^{2} \binom{n}{j}(m-1)^j + \binom{n-1}{2}(m-1)^3.$$

Let $2+r$ be the maximum weight of a word in C, $1 \leq r \leq 3$. For $r \leq 2$ the upper bound for $|C|$ follows by Lemma 4.2.
Therefore we assume $r = 3$.
By Theorem 4.1 there are positions i_1, i_2, i_3, k_1, k_2 and nonzero integers a_{i_1}, a_{i_2}, a_{i_3}, w.l.o.g. $i_1 = 1, i_2 = 2, i_3 = 3, k_1 = 1, k_2 = 2$ and $a_{i_j} = 1$ for $1 \leq j \leq 3$, and also 6 words $u^{(1)}, ..., u^{(6)}$ of weight 5 and 6 words $v^{(1)}, ..., v^{(6)}$ of weight 4 such that

$$u^{(j)}(1) = 1, u^{(j)}(2) = 1, u^{(j)}(3) = 1$$

for every j, $1 \leq j \leq 6$ and

$$v^{(j)}(1) = 1, v^{(j)}(2) = 1$$

for every j, $1 \leq j \leq 6$. Also the supports of $u^{(1)}, ..., u^{(6)}$ intersect exactly in positions $1, 2, 3$ and the supports of $v^{(1)}, ..., v^{(6)}$ intersect exactly in positions $1, 2$.

We denote by A the number of words in C of weight at least 1. We know $\overline{0} \in C$, therefore

$$|C| = A + 1.$$

Let A'_s be the number of all words in C of weight $2+s$, $-1 \leq s \leq 3$. Then we have

$$A = \sum_{s=-1}^{3} A'_s. \tag{4.28}$$

By Theorem 4.2 we have for $n \geq n_0$

$$A'_3 \leq \binom{n-3}{2}(m-1)^3.$$

Now we show that

$$A'_2 \leq \binom{n-2}{2}(m-1)^2 + 2(m-1)^2(m-2)(n-3). \tag{4.29}$$

Let $v \in C$ be a word of weight 4. By Corollary 4.1(2) we have

$$\{1,2\} \subseteq supp(v).$$

If $m = 2$ then for every word v of weight 4 in C we have

$$v(1) = 1, v(2) = 1.$$

Thus for every $m = 2$ we conclude

$$A'_2 \leq \binom{n-2}{2},$$

which confirms (4.29).
Therefore we consider $m \geq 3$.
In the notation of Corollary 4.1(2) we have $s = 2$, $k = 0$ and $X' \leq 1$. This means for every word $v \in C$ of weight 4 holds

$$v(1) = 1, v(2) = 1$$

except possibly for one position $j \in \{1,2\}$, where we may have a nonzero entry $v(j) \neq 1$.
The number of words v of weight 4 with both first entries equal to 1 in C is at most

$$\binom{n-2}{2}(m-1)^2. \tag{4.30}$$

Now consider $v \in C$ of weight 4 such that $v(2) = c \neq 0, 1$. We show that $v(3) \neq 0$.
Let $B = \{1,2,3\}$ and $B' = \{1,...,n\} \setminus B$. Suppose $v(3) = 0$ then for every i, $1 \leq i \leq 6$, we have $d(v|B, u^{(i)}|B) = 2$. As $w(v) < 6$, there exists i, $1 \leq i \leq 6$, such that

$$supp(v|B') \cap supp(u^{(i)}|B') = \phi.$$

Then we have

$$d(v|B', u^{(i)}|B') = 4,$$

thus

$$d(v, u^{(i)}) = d(v|B, u^{(i)}|B) + d(v|B', u^{(i)}|B') = 2 + 4 = 6.$$

But v and $u^{(i)}$ are in C with $d(v, u^{(i)}) \leq 5$ and it is a contradiction. Therefore the number of these words in C is at most

$$(m-2)(m-1)\binom{n-3}{1}(m-1) = (m-1)^2(m-2)(n-3).$$

Similarly, if $v \in C$, $v(1) = c \neq 0, 1$ we have the same result. Therefore the number of words v of weight 4 in C, which have not both entries $v(1) = v(2) = 1$, is at most

$$2(m-1)^2(m-2)(n-3). \tag{4.31}$$

By (4.30),(4.31) we conclude

$$A_2' \leq \binom{n-2}{2}(m-1)^2 + 2(m-1)^2(m-2)(n-3)$$

which confirms (4.29).

For every $n \geq n_0$, $m \geq 2$, we have

$$\begin{aligned} A_2' &\leq \binom{n-2}{2}(m-1)^2 + 2(m-1)^2(m-2)(n-3) \\ &\leq \binom{n-2}{2}(m-1)^2 + 2(n-3)(m-1)^3 - 2(n-3)(m-1)^2 \\ &\leq 2(n-3)(m-1)^3 + \left[\binom{n-2}{2} - 2(n-3)\right](m-1)^2. \end{aligned}$$

For $s = 1$ by Corollary 4.1(1), every word v of weight 3, must have

$$|supp(v) \cap \{1, 2, 3\}| \geq \frac{r+s}{2} = \frac{3+1}{2} = 2$$

and also $0 \leq k \leq 1$ and $X \leq 2k + 1$.
The number of words v of weight 3 with $|supp(v) \cap \{1, 2, 3\}| = 2$ and $X = 0$ in C is at most

$$\binom{3}{2}\binom{n-3}{1}(m-1). \tag{4.32}$$

Now suppose $|supp(v) \cap \{1, 2, 3\}| = 2$ and $X = 1$. We show that $v(3) \neq 1$. If $v(3) = 1$ then $v(1) \neq 0, 1$ or $v(2) \neq 0, 1$, w.l.o.g. suppose $v(1) \neq 0, 1$. Let $B = \{1, 2, 3\}$ and $B' = \{1, ..., n\} \setminus B$. Consider the words $v^{(1)}, ..., v^{(6)}$ of weight 4. We know the supports of these words are disjoint in positions

$3, ..., n$. Therefore there are at least 5 words among these words, w.l.o.g. $v^{(1)}, ..., v^{(5)}$, such that $v^{(j)}(3) = 0$ for every j, $1 \leq j \leq 5$. Thus we have

$$d(v|B, v^{(j)}|B) = 3$$

for every j, $1 \leq j \leq 5$. As $w(v) < 5$, there exist j, $1 \leq j \leq 5$, such that

$$supp(v|B') \cap supp(v^{(j)}|B') = \phi.$$

Therefore

$$d(v|B', v^{(j)}|B') = 3$$

and we have

$$d(v, v^{(j)}) = d(v|B, v^{(j)}|B) + d(v|B', v^{(j)}|B') = 3 + 3 = 6.$$

But $v, v^{(j)} \in C$ with $d(v, v^{(j)} \leq 5$ and it is a contradiction.
Therefore for every word $v \in C$ of weight 3 with $|supp(v) \cap \{1, 2, 3\}| = 2$ and $X = 1$ we have $v(3) \neq 1$. The number of these words with $v(3) = 0$ in C is at most

$$\binom{2}{1}(m-2)\binom{n-3}{1}(m-1)$$

and similarly, we have the same result for $v(3) = c \neq 0, 1$.
Therefore the number of words v of weight 3 with $|supp(v) \cap \{1, 2, 3\}| = 2$ and $X = 1$ in C is at most

$$2\binom{2}{1}(m-2)\binom{n-3}{1}(m-1). \tag{4.33}$$

Also the number of words v of weight 3 with $|supp(v) \cap \{1, 2, 3\}| = 3$ in C is at most

$$(m-1)^3. \tag{4.34}$$

We have by (4.32),(4.33),(4.34)

$$\begin{aligned} A'_1 &\leq \binom{3}{2}\binom{n-3}{1}(m-1) + 2\binom{2}{1}(m-2)\binom{n-3}{1}(m-1) + (m-1)^3 \\ &\leq 3(n-3)(m-1) + 4(n-3)(m-1)^2 - 4(n-3)(m-1) + (m-1)^3 \\ &\leq (m-1)^3 + 4(n-3)(m-1)^2 + (-n+3)(m-1). \end{aligned}$$

For $s = 0$, by Corollary 4.1(2), every word v of weight 2 in C, must have

$$|supp(v) \cap \{1, 2\}| \geq \frac{r+s-1}{2} = \frac{3+0-1}{2} = 1$$

and also $0 \leq k \leq 1$ and $X' \leq 2k+1$.
The number of words v of weight 2 with $|supp(v) \cap \{1,2\}| = 1$ and $X' = 0$ in C is at most
$$\binom{2}{1}\binom{n-2}{1}(m-1). \tag{4.35}$$
Now suppose $|supp(v) \cap \{1,2\}| = 1$ and $X' = 1$. We show that $v(3) \neq 0$.
Let $B = \{1,2,3\}$ and $B' = \{1,...,n\} \setminus B$. If $v(3) = 0$ then

$$d(v|B, u^{(j)}|B) = 3$$

for every j, $1 \leq j \leq 6$. As $w(v) < 6$, there exist j, $1 \leq j \leq 6$, such that

$$supp(v|B') \cap supp(u^{(j)}|B') = \phi.$$

Therefore
$$d(v|B', u^{(j)}|B') = 3$$
and we have

$$d(v, u^{(j)}) = d(v|B, u^{(j)}|B) + d(v|B', u^{(j)}|B') = 3 + 3 = 6.$$

But $v, u^{(j)} \in C$ with $d(v, u^{(j)}) \leq 5$ and it is a contradiction.
Therefore for every word v of weight 2 with $|supp(v) \cap \{1,2\}| = 1$ and $X' = 1$ we have $v(3) \neq 0$. The number of these words in C is at most

$$\binom{2}{1}(m-2)(m-1). \tag{4.36}$$

Also the number of words v of weight 2 with $|supp(v) \cap \{1,2\}| = 2$ in C is at most
$$(m-1)^2. \tag{4.37}$$
We have by (4.35),(4.36),(4.37)

$$\begin{aligned} A'_0 &\leq \binom{2}{1}\binom{n-2}{1}(m-1) + \binom{2}{1}(m-2)(m-1) + (m-1)^2 \\ &\leq 2(n-2)(m-1) + 2(m-1)^2 - 2(m-1) + (m-1)^2 \\ &\leq 3(m-1)^2 + (2n-6)(m-1). \end{aligned}$$

For $s = -1$, by Corollary 4.1(1), every word v of weight 1, must have

$$|supp(v) \cap \{1,2,3\}| \geq \frac{r+s}{2} = \frac{3-1}{2} = 1.$$

Therefore the number of these words in C is at most

$$A'_{-1} \leq 3(m-1).$$

We conclude by (4.28)

$$\begin{aligned} A \;\leq\; & \left[\binom{n-3}{2} + 2(n-3) + 1\right](m-1)^3 + \\ & \left[\binom{n-2}{2} - 2(n-3) + 4(n-3) + 3\right](m-1)^2 + \\ & \left[-n+3+2n-6+3\right](m-1) \end{aligned}$$

and

$$\begin{aligned} A \;\leq\; & \left[\binom{n-3}{2} + 2(n-3) + 1\right](m-1)^3 + \\ & \left[\binom{n-2}{2} + 2(n-3) + 3\right](m-1)^2 + \\ & n(m-1). \end{aligned}$$

We know

$$\begin{gathered} \binom{n-3}{2} + 2(n-3) + 1 \\ = \\ \binom{2}{0}\binom{n-3}{2} + \binom{2}{1}\binom{n-3}{1} + \binom{2}{2}\binom{n-3}{0} \\ = \\ \binom{n-1}{2}, \end{gathered}$$

where the last equation follows by Lemma 2.1(1).
Similarly we have

$$\begin{gathered} \binom{n-2}{2} + 2(n-3) + 3 \\ = \\ \binom{n-2}{2} + 2(n-2) - 2 + 3 \\ = \\ \binom{2}{0}\binom{n-2}{2} + \binom{2}{1}\binom{n-2}{1} + \binom{2}{2}\binom{n-2}{0} = \binom{n}{2}. \end{gathered}$$

Therefore A can be estimated by

$$A \leq \binom{n-1}{2}(m-1)^3 + \binom{n}{2}(m-1)^2 + n(m-1).$$

We conclude

$$|C| = A + 1 \leq \sum_{j=0}^{2} \binom{n}{j}(m-1)^j + \binom{n-1}{2}(m-1)^3 = \omega_0(m,n,5).$$

$\square$

The ω_0-conjecture for $HG(m,n,d)$ is trivially true for distance $d=1$. By Corollary 3.3, Corollary 4.2 and Proposition 4.1 this conjecture has now been proved for $HG(m,n,d)$ up to distance $d=6$.

Chapter 5

Binary Hamming Graphs

In this chapter we concentrate on Hamming graphs $HG(m,n,d)$ with $m=2$. We will prove the ω_0-conjecture for these graphs for arbitrary distance $d \geq 2$.

First we solve the even distance case, $d = 2t$.

Proposition 5.1. *For every positive integer t there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $n \in \mathbb{N}$,*

$$\omega(HG(2,n,2t)) = \sum_{j=0}^{t} \binom{n}{j}.$$

Proof. We know for $n \geq 2t$ by Proposition 1.2

$$\omega(HG(2,n,2t)) \geq \sum_{j=0}^{t} \binom{n}{j}.$$

Let C be a maximal clique of $HG(2,n,2t)$. By Proposition 1.1 we may assume $\overline{0} \in C$. Therefore for every word $u \in C$, we have $w(u) \leq 2t$. We show that

$$|C| \leq \sum_{j=0}^{t} \binom{n}{j},$$

if n is sufficiently large.
Let $t+r$, $0 \leq r \leq t$, be the maximum weight of a word in C. As for $r=0$ the upper bound is trivially true, we assume $r \geq 1$.
Let n_0 be chosen according to Theorem 3.1 and $n \geq n_0$. By Theorem 3.1 and Corollary 3.1 there are positions $i_1, ..., i_r$, w.l.o.g. $i_1 = 1, ..., i_r = r$, such that for every word $u \in C$ of weight $t+r$ we have

$$u(1) = u(2) = ... = u(r) = 1.$$

Let A_1 be the number of all words of weight at most $t-r$. These words have at most distance $2t$ to every word in C. They must be contained in C, because C is maximal. We have

$$A_1 = \sum_{j=0}^{t-r} \binom{n}{j}.$$

If we denote by A_2 the number of words in C of weight at least $t-r+1$ then we have

$$|C| = A_1 + A_2 = \sum_{j=0}^{t-r} \binom{n}{j} + A_2. \tag{5.1}$$

According to Corollary 3.1 for $n \geq n_0$ every word in C of weight $t+s$, $-r+1 \leq s \leq r$, must have at least $\lceil \frac{r+s}{2} \rceil \geq 1$ entries 1 among the first r positions. Let $B_{k,s}$ be the number of words of weight $t+s$ in C which have exactly $\lceil \frac{r+s}{2} \rceil + k$ entries 1 among positions $1, ..., r$. We have

$$B_{k,s} \leq \binom{r}{\lceil \frac{r+s}{2} \rceil + k} \binom{n-r}{t+s-\lceil \frac{r+s}{2} \rceil - k},$$

$$A_2 = \sum_{s=-r+1}^{r} \sum_{k=0}^{r-1} B_{k,s} = \sum_{k=0}^{r-1} \sum_{s=-r+1}^{r} B_{k,s},$$

and

$$A_2 \leq \sum_{k=0}^{r-1} B_k,$$

where

$$B_k = \sum_{s=-r+1}^{r} \binom{r}{\lceil \frac{r+s}{2} \rceil + k} \binom{n-r}{t+s-\lceil \frac{r+s}{2} \rceil - k}. \tag{5.2}$$

We set $s = -r + e$ $(1 \leq e \leq 2r)$, thus $\lceil \frac{r+s}{2} \rceil = \lceil \frac{e}{2} \rceil$ and (5.2) can be written as

$$B_k = \sum_{e=1}^{2r} \binom{r}{\lceil \frac{e}{2} \rceil + k} \binom{n-r}{t-r+e-\lceil \frac{e}{2} \rceil - k}.$$

We split this sum in two parts:

$$B_k = \sum_{1 \leq e \leq 2r,\ e \text{ odd}} \binom{r}{\lceil \frac{e}{2} \rceil + k} \binom{n-r}{t-r+e-\lceil \frac{e}{2} \rceil - k}$$

$$+$$

$$\sum_{1\le e\le 2r,\ e \text{ odd}} \binom{r}{\lceil \frac{e+1}{2}\rceil + k}\binom{n-r}{t-r+e+1-\lceil\frac{e+1}{2}\rceil - k}. \tag{5.3}$$

If $e = 2j+1$ $(0 \le j \le r-1)$ then $\lceil \frac{e}{2}\rceil = \lceil\frac{e+1}{2}\rceil = j+1$ and (5.3) can be written as

$$B_k = \sum_{j=0}^{r-1}\binom{r}{j+1+k}\binom{n-r}{t-r+j-k}$$
$$+$$
$$\sum_{j=0}^{r-1}\binom{r}{j+1+k}\binom{n-r}{t-r+j+1-k}.$$

Substitute $j' = j+1+k$, $1+k \le j' \le r+k$. Applying $\binom{r}{j'} = 0$ for $j' > r$ we get

$$B_k = \sum_{j'=1+k}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'} + \sum_{j'=1+k}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}. \tag{5.4}$$

The first sum in (5.4) can be transformed to

$$\sum_{j'=1+k}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'}$$
$$=$$
$$\sum_{j'=0}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'} - \sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'}.$$

By Proposition 2.1(3) we have

$$\sum_{j'=0}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'} = \sum_{j'=0}^{r}\binom{r}{r-j'}\binom{n-r}{t-2k-1-(r-j')}.$$

Substitute $p = r-j'$, $0 \le p \le r$. By Lemma 2.1(1), we have

$$\sum_{p=0}^{r}\binom{r}{p}\binom{n-r}{t-2k-1-p} = \binom{n}{t-2k-1}.$$

If the second sum in (5.4) is transformed in the same way, then B_k can be written as

$$\binom{n}{t-2k-1} - \sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k-1+j'}$$

$$+$$

$$\binom{n}{t-2k} - \sum_{j'=0}^{k} \binom{r}{j'} \binom{n-r}{t-r-2k+j'}.$$

Change parameter j' to j. Then we have

$$A_2 \le \sum_{k=0}^{r-1} B_k = \sum_{k=0}^{r-1} \left[\binom{n}{t-2k} + \binom{n}{t-2k-1} \right] - a,$$

$$A_2 \le \sum_{j=t-2r+1}^{t} \binom{n}{j} - a, \tag{5.5}$$

where

$$a = \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r}{t-r-2k-1+j} + \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r}{t-r-2k+j}.$$

By Proposition 2.1(2), a can be written as

$$\begin{aligned} a &= \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \left[\binom{n-r}{t-r-2k-1+j} + \binom{n-r}{t-r-2k+j} \right] \\ &= \sum_{k=0}^{r-1} \sum_{j=0}^{k} \binom{r}{j} \binom{n-r+1}{t-r-2k+j} \end{aligned}$$

and Lemma 2.4 implies

$$a = \sum_{j=t-2r+1}^{t-r} \binom{n}{j}.$$

Inserting in (5.5) yields

$$A_2 \le \sum_{j=t-2r+1}^{t} \binom{n}{j} - \sum_{j=t-2r+1}^{t-r} \binom{n}{j} = \sum_{j=t-r+1}^{t} \binom{n}{j}.$$

Finally, we conclude by (5.1)

$$|C| \le \sum_{j=0}^{t-r} \binom{n}{j} + \sum_{j=t-r+1}^{t} \binom{n}{j} = \sum_{j=0}^{t} \binom{n}{j}.$$

$\square$

Now we solve the odd distance case, $d = 2t+1$.

Proposition 5.2. *For every positive integer t there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $n \in \mathbb{N}$,*

$$\omega(HG(2,n,2t+1)) = \sum_{j=0}^{t} \binom{n}{j} + \binom{n-1}{t}.$$

Proof. We know for $n \geq 2t+1$ by Proposition 1.2

$$\omega(HG(2,n,2t+1)) \geq \sum_{j=0}^{t} \binom{n}{j} + \binom{n-1}{t}.$$

Let C be a maximal clique of $HG(2,n,2t+1)$. By Proposition 1.1 we may assume $\overline{0} \in C$. Therefore for every word $u \in C$, we have $w(u) \leq 2t+1$. We show that

$$|C| \leq \sum_{j=0}^{t} \binom{n}{j} + \binom{n-1}{t} \tag{5.6}$$

if n is sufficiently large.
Let $t+r$, $1 \leq r \leq t+1$, be the maximum weight of a word in C. Let n_0, positions $i_1, ..., i_r, k_1, ..., k_{r-1}$ and words

$$u^{(1)}, ..., u^{(2t+2)}, v^{(1)}, ..., v^{(2t+2)}$$

be determined as in Theorem 4.1. W.l.o.g. we may assume $i_1 = 1, ..., i_r = r, k_1 = 1, ..., k_{r-1} = r-1$.
If $r = 1$ then we have by Corollary 4.1(1), $u(1) = 1$ for every word $u \in C$ of weight $t+1$. In this case the estimate (5.6) becomes obvious.
Therefore we consider $2 \leq r \leq t+1$. Let A_1 be the number of all words of weight at most $t-r+1$. These words have at most distance $2t+1$ to every word in C. They must be contained in C, because C is maximal. We have

$$A_1 = \sum_{j=0}^{t-r+1} \binom{n}{j}.$$

If we denote by A_2 the number of words in C of weight at least $t-r+2$ then we have

$$|C| = A_1 + A_2 = \sum_{j=0}^{t-r+1} \binom{n}{j} + A_2. \tag{5.7}$$

According to Corollary 4.1(1), every word in C of weight $t+s,\ s=-r+2j\ (1\leq j\leq r)$, must have at least j entries 1 among the first r positions. Let $B_{k,s}$ be the number of words of weight $t+s,\ s=-r+2j\ (1\leq j\leq r)$, in C which have exactly $j+k$ entries 1 among positions $1,...,r$. We have

$$B_{k,s}\leq\binom{r}{j+k}\binom{n-r}{t+s-j-k}.$$

Also according to Corollary 4.1(2), every word in C of weight $t+s,\ s=-r+2j+1\ (1\leq j\leq r-1)$, must have at least j entries 1 among the first $r-1$ positions. Let $B'_{k,s}$ be the number of words of weight $t+s,\ s=-r+2j+1\ (1\leq j\leq r-1)$, in C which have exactly $j+k$ entries 1 among positions $1,...,r-1$. We have

$$B'_{k,s}\leq\binom{r-1}{j+k}\binom{n-(r-1)}{t+s-j-k}.$$

Therefore we have

$$\begin{aligned}A_2 &= \sum_{s=-r+2j\ (1\leq j\leq r)}\sum_{k=0}^{r-1}B_{k,s}+\sum_{s=-r+2j+1\ (1\leq j\leq r-1)}\sum_{k=0}^{r-2}B'_{k,s},\\ &= \sum_{k=0}^{r-1}\sum_{s=-r+2j\ (1\leq j\leq r)}B_{k,s}+\sum_{k=0}^{r-2}\sum_{s=-r+2j+1\ (1\leq j\leq r-1)}B'_{k,s},\end{aligned}$$

$$A_2\leq$$

$$\sum_{k=0}^{r-1}\sum_{s=-r+2j\ (1\leq j\leq r)}\binom{r}{j+k}\binom{n-r}{t+s-j-k}$$
$$+$$
$$\sum_{k=0}^{r-2}\sum_{s=-r+2j+1\ (1\leq j\leq r-1)}\binom{r-1}{j+k}\binom{n-(r-1)}{t+s-j-k}. \tag{5.8}$$

The first sum in (5.8) can be written as

$$\sum_{k=0}^{r-1}\sum_{j=1}^{r}\binom{r}{j+k}\binom{n-r}{t-r+2j-j-k}. \tag{5.9}$$

Also the second sum in (5.8) can be written as

$$\sum_{k=0}^{r-2}\sum_{j=1}^{r-1}\binom{r-1}{j+k}\binom{n-(r-1)}{t-r+2j+1-j-k}. \tag{5.10}$$

Therefore (5.8),(5.9),(5.10) imply

$$A_2 \leq$$

$$\sum_{k=0}^{r-1}\sum_{j=1}^{r}\binom{r}{j+k}\binom{n-r}{t-r+j-k}+\sum_{k=0}^{r-2}\sum_{j=1}^{r-1}\binom{r-1}{j+k}\binom{n-(r-1)}{t-r+j+1-k}. \tag{5.11}$$

In (5.11) consider the first sum. Substitute $j' = j+k$, $1+k \leq j' \leq r+k$. Applying $\binom{r}{j'} = 0$ for $j' > r$ we get

$$\sum_{j=1}^{r}\binom{r}{j+k}\binom{n-r}{t-r+j-k}$$

$$=$$

$$\sum_{j'=1+k}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}$$

$$=$$

$$\sum_{j'=0}^{r}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}-\sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}$$

$$=$$

$$\sum_{j'=0}^{r}\binom{r}{r-j'}\binom{n-r}{t-2k-(r-j')}-\sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}. \tag{5.12}$$

Substitute $p = r - j'$, $0 \leq p \leq r$. Equation (5.12) can be transformed to

$$\sum_{p=0}^{r}\binom{r}{p}\binom{n-r}{t-2k-p}-\sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k+j'},$$

and by Lemma 2.1(1), we have

$$\binom{n}{t-2k}-\sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}.$$

Therefore the first sum in (5.11) can be transformed to

$$\sum_{k=0}^{r-1}\left[\binom{n}{t-2k}-\sum_{j'=0}^{k}\binom{r}{j'}\binom{n-r}{t-r-2k+j'}\right]. \tag{5.13}$$

Similarly the second sum in (5.11) can be transformed to

$$\sum_{k=0}^{r-2}\left[\binom{n}{t-2k}-\sum_{j'=0}^{k}\binom{r-1}{j'}\binom{n-(r-1)}{t-r-2k+1+j'}\right]. \tag{5.14}$$

Change parameter j' to j. Then by (5.13), (5.14), equation (5.11) can be written as

$$\begin{aligned} A_2 &\le \sum_{k=0}^{r-1}\binom{n}{t-2k}+\sum_{k=0}^{r-2}\binom{n}{t-2k}-(a+b) \\ &\le 2\sum_{k=0}^{r-2}\binom{n}{t-2k}+\binom{n}{t-2(r-1)}-(a+b), \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} a &= \sum_{k=0}^{r-1}\sum_{j=0}^{k}\binom{r}{j}\binom{n-r}{t-r-2k+j} \\ b &= \sum_{k=0}^{r-2}\sum_{j=0}^{k}\binom{r-1}{j}\binom{n-(r-1)}{t-r-2k+1+j}. \end{aligned}$$

We have by Lemma 2.5

$$a+b=\sum_{j=t-2r+1}^{t-r+1}\binom{n}{j}-\binom{n-1}{t-2r}-\binom{n}{t-2r+2}.$$

Inserting $a+b$ in (5.15) yields

$$A_2 \le 2\sum_{k=0}^{r-1}\binom{n}{t-2k}+\binom{n-1}{t-2r}-\sum_{j=t-2r+1}^{t-r+1}\binom{n}{j}.$$

Applying Lemma 2.1(4) to the first sum shows

$$A_2 \le \sum_{j=t-2r+1}^{t}\binom{n}{j}+\binom{n-1}{t}-\binom{n-1}{t-2r}+\binom{n-1}{t-2r}-\sum_{j=t-2r+1}^{t-r+1}\binom{n}{j}.$$

We replace the last sum by

$$\sum_{j=t-2r+1}^{t-r+1}\binom{n}{j}=\sum_{j=0}^{t-r+1}\binom{n}{j}-\sum_{j=0}^{t-2r}\binom{n}{j}$$

and achieve

$$A_2 \leq \sum_{j=0}^{t} \binom{n}{j} + \binom{n-1}{t} - \sum_{j=0}^{t-r+1} \binom{n}{j}.$$

Finally, we conclude by (5.7)

$$|C| = \sum_{j=0}^{t-r+1} \binom{n}{j} + A_2 \leq \sum_{j=0}^{t} \binom{n}{j} + \binom{n-1}{t}.$$

$\square$

Chapter 6

Further Results

In this chapter we present further special values for the clique number and for the chromatic number of $HG(m, n, d)$ as well as for its complement $\overline{HG}(m, n, d)$.

First we begin by recalling the necessary definitions and notations.
The *complement* of a simple graph G, written $\overline{G}$, is a graph having the same vertex set as G, such that u, v are adjacent in $\overline{G}$ if and only if u, v are not adjacent in G. A *subgraph* of graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, we write this as $H \subseteq G$. An *induced subgraph* of G is a subgraph H such that every edge of G with end points in $V(H)$ belongs to $E(H)$. If H is an induced subgraph of G with vertex set S, then we write $H = G[S]$.
An *independent set* in a graph G is a vertex subset $S \subseteq V(G)$ such that the induced subgraph $G[S]$ has no edges. We use $\alpha(G)$ to denote the size of the largest independent set in G, $\alpha(G) = \omega(\overline{G})$.
A *proper coloring* of a graph G is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is defined as the minimum n for which G has a proper n-coloring. Simple bounds on $\chi(G)$ include $\chi(G) \leq |V(G)|$, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

Lemma 6.1. *Let $m, n, d, m \geq 2, n \geq d$, be positive integers. Then we have*

$$\omega(HG(m, n, d)) \geq m^d.$$

Proof. Consider set

$$C = \{(x_1, ..., x_d, 0, ..., 0) : x_i \in \mathbb{Z}_m, 1 \leq i \leq d\}.$$

For every $x, x' \in C$, we have $d(x, x') \leq d$. Thus $HG(m, n, d)[C]$ is a complete subgraph and $|C| = m^d$. Therefore we have

$$\omega(HG(m, n, d)) \geq |C| = m^d.$$

□

Theorem 6.1.

1. *For $n > m \geq 2$ we have*

$$\omega(HG(m, n, 2)) = \omega_0(m, n, 2) = n(m - 1) + 1$$

2. *For $m + 1 \geq n \geq 2$ we have*

$$\omega(HG(m, n, 2)) = m^2.$$

Proof.

1. We know by Proposition 1.2

$$\omega(HG(m, n, 2)) \geq \omega_0(m, n, 2) = \sum_{j=0}^{1} \binom{n}{j} (m - 1)^j = n(m - 1) + 1.$$

Let C be a maximal clique of $HG(m, n, 2)$. We show that

$$|C| \leq n(m - 1) + 1.$$

By Proposition 1.1 we may assume $\overline{0} \in C$. Therefore for every word $u \in C$, we have weight $w(u) \leq 2$. If for every $v \in C$, $w(v) \leq 1$ then the upper bound is trivially true.

So we may assume that there is a word $u \in C$, $w(u) = 2$, w.l.o.g.

$$u = (1, 1, 0, ..., 0).$$

Every word $v \in C$ of weight 1 has $v(1) \neq 0$ or $v(2) \neq 0$. Suppose that for every $v \in C$ of weight 2 holds $v(3) = ... = v(n) = 0$ then

$$|C| \leq m^2 = (m + 1)(m - 1) + 1 \leq n(m - 1) + 1.$$

So we may assume that there is a word $v \in C$ of weight 2 with $v(i) \neq 0$ for some $i \geq 3$, w.l.o.g. $v(3) = 1$. As $d(u, v) \leq 2$ implies $v(1) = 1$ or $v(2) = 1$, we have $v(4) = ... = v(n) = 0$.

Therefore we may assume $v = (1, 0, 1, 0, ..., 0)$. Now we have the following 3 words in C:

$$\begin{array}{rcl} \overline{0} & = & (0, 0, 0, 0, ..., 0) \\ u & = & (1, 1, 0, 0, ..., 0) \\ v & = & (1, 0, 1, 0, ..., 0). \end{array}$$

But $n(m-1)+1 \geq 4$ and there exists a word $z \in C$, $z \notin \{\overline{0}, u, v\}$. Let $B = \{1, 2\}$ and $B' = \{1, ..., n\} \setminus B$. Consider the following cases.

Case 1: $z(1) = 0$ for a word $z \in C$, $z \notin \{\overline{0}, u, v\}$

If $z(2) = 0$ then $d(u|B, z|B) = 2$ implies $z = 0$ which is a contradiction.
If $z(2) > 1$ then $d(u|B, z|B) = 2$ implies $z(3) = 0$ and $d(v, z) \geq 3$, which is a contradiction.
So we must have $z(2) = 1$. From $d(v|B, z|B) = 2$ we conclude $z(i) = v(i)$ for $i \in B'$, i.e. $z(3) = 1, z(4) = ... = z(n) = 0$,

$$z = (0, 1, 1, 0, ..., 0).$$

We show $C = \{\overline{0}, u, v, z\}$. Suppose $s \in C$, $s \notin \{\overline{0}, u, v, z\}$. We know $s(1) \neq 0$, because otherwise $s = z$. Suppose $s(2) \neq 1$. Then we have $d(s|B, z|B) = 2$ and $s(i) = z(i)$ for $i \in B'$, i.e. $s(3) = 1, s(4) = ... = s(n) = 0$, $s = (s_1, s_2, 1, 0, ..., 0)$. But $w(s) \leq 2$ requires $s(2) = 0$. Now $d(s, u) = 2$ implies $s(1) = u(1) = 1$ and $s = (1, 0, 1, 0, ..., 0) = v$, which is a contradiction.
So we have $s(2) = 1$. As $s(1) \neq 0$ and $w(s) \leq 2$ we conclude $s(3) = ... = s(n) = 0$, i.e. $s = (s_1, 1, 0, ..., 0)$. But $s \neq u$ implies $s(1) \geq 2$ and we have $d(s, v) = 3$, which is a contradiction.
Therefore we have $C = \{\overline{0}, u, v, z\}$ and $4 = |C| \leq n(m-1)+1$.

Case 2: $z(1) \neq 0$ for every $z \in C$, $z \notin \{\overline{0}, u, v\}$

Suppose $z \in C$, $z \notin \{\overline{0}, u, v\}$, $z(1) = a > 1$. If $z(2) \neq 0$ then $d(z|B, v|B) = 2$ and we conclude $z(i) = v(i)$ for $i \in B'$, especially $z(3) = v(3) = 1$. But then we have $w(z) \geq 3$, which is a contradiction.
Therefore we have $z(2) = 0$. As $d(z|B, u|B) = 2$, we conclude $z(3) = ... = z(n) = 0$ and $z = (a, 0, ..., 0), a \neq 0, 1$.
Now C may contain the following types of words:
1) $\overline{0} = (0, ..., 0)$,
2) words $(a, 0, ..., 0)$ of weight 1, $a > 1$, the number of these words is $m-2$,
3) words of weight at most 2 and first entry 1, the number of these words is

$(n-1)(m-1)+1$.
Summing up yields

$$|C| \leq 1 + m - 2 + (n-1)(m-1) + 1 = n(m-1) + 1.$$

2. By Lemma 6.1 we have $\omega(HG(m,n,2)) \geq m^2$. To proof the upper bound suppose that C is a clique of size $|C| > m^2$, $\overline{0} \in C$. The number of words of weight at most 1 in C is at most $n(m-1)+1$. We have

$$n(m-1) + 1 \leq (m+1)(m-1) + 1 = m^2 < |C|.$$

Therefore C has at least one word u of weight 2, w.l.o.g. $u = (1,1,0,...,0)$. Suppose that for every $x \in C$ holds $x_3 = ... = x_n = 0$ then $|C| \leq m^2$ and it is contradiction.
Similar to part 1 we may assume that C contains the words

$$\overline{0} = (0,0,0,0,...,0), u = (1,1,0,0,...,0), v = (1,0,1,0,...,0).$$

As $|C| > m^2 \geq 4$, there exist a word $z \in C$, $z \notin \{\overline{0}, u, v\}$. Case 1 of part 1 implies $|C| = 4 \leq m^2$ contradicting $|C| > m^2$. By Case 2 of part 1 we conclude $|C| \leq n(m-1)+1 < m^2$, which is also a contradiction. $\square$

Lemma 6.2. *Let C be a clique in $HG(2,n,3)$. Then for $n \geq 4$ the maximum number of words of weight 2 in C is $n-1$.*

Proof. We know any two words of weight 2 must overlap in an entry 1. Consider words $u, v \in C$ of weight 2. We may suppose

$$u = (1,1,0,...,0), \ v = (1,0,1,0,...,0).$$

Assume now that there is a word $z \in C$ of weight 2 with first entry 0. Then the entries in positions 2 and 3 must be 1 to overlap with u and v, therefore $z = (0,1,1,0,...,0)$. No other word of weight 2 is adjacent to u, v, z. So the number of these words is at most $3 \leq n-1$ for $n \geq 4$.
Now suppose that every word of weight 2 in C has entry 1 in the first position. Then the number of these words in C is at most $n-1$.

$\square$

Theorem 6.2. *For every $n \geq 4$ we have*

$$\omega(HG(2,n,3)) = \omega_0(2,n,3) = 2n.$$

Proof. We know by Proposition 1.2

$$\omega(HG(2,n,3)) \geq \sum_{j=0}^{1} \binom{n}{j} + \binom{n-1}{1} = 2n.$$

Let C be a maximal clique of $HG(2,n,3)$. By Proposition 1.1 we may assume $\overline{0} \in C$. Therefore for every word $u \in C$, we have $w(u) \leq 3$. We show that $|C| \leq 2n$ for $n \geq 4$. Consider the following cases.

Case 1: The maximum weight of a word in C is equal to 2

By Lemma 6.2 we have at most $n-1$ words of weight 2 in C. There are n words of weight 1 and the zero word in C. That gives

$$|C| \leq (n-1) + n + 1 = 2n.$$

Case 2: The maximum weight of a word in C is equal to 3

Observe that any two words of weight 3 overlap in two entries 1. Also every word of weight 1 shares its entry 1 with every word of weight 3.

Case 2.1: C has exactly one word of weight 3

We may assume $u = (1,1,1,0,...,0) \in C$. Then C contains $\overline{0}$, three words of weight 1, at most $n-1$ words of weight two by Lemma 6.2 and u. That yields

$$|C| \leq 1 + 3 + n - 1 + 1 = n + 4 \leq 2n \text{ for } n \geq 4.$$

Case 2.2: C has exactly two words of weight 3

We may assume $u = (1,1,1,0,...,0)$, $v = (1,1,0,1,0,...,0) \in C$. Now C contains only two words of weight 1 and we have

$$|C| \leq 1 + 2 + n - 1 + 2 = n + 4 \leq 2n \text{ for } n \geq 4.$$

Case 2.3: C contains at least 3 words of weight 3

Case 2.3.1: C contains 3 words u, v, z of weight 3 with a common pair of positions in their supports

W.l.o.g. we may assume

$$\begin{aligned} u &= (1,1,1,0,0,0,...,0), \\ v &= (1,1,0,1,0,0,...,0), \\ z &= (1,1,0,0,1,0,...,0). \end{aligned}$$

In this case we have $n \geq 5$. Every word of weight 3 must have first and second entry equal to 1. So the number of words of weight 3 in C is at most $n-2$. Moreover, we have in C the zero word, two words of weight 1 and at most $n-1$ words of weight 2 by Lemma 6.2 That yields

$$|C| \leq 1 + 2 + n - 1 + n - 2 = 2n.$$

Case 2.3.2: No three words of weight 3 in C have a common pair of positions in their supports

Let u, v, z be words of weight 3 in C. We now may assume

$$\begin{aligned} u &= (1,1,1,0,0,...,0), \\ v &= (1,1,0,1,0,...,0), \\ z &= (1,0,1,1,0,...,0). \end{aligned}$$

If C contains no other word of weight 3 then we have in C the zero word, one word of weight 1, at most $n-1$ words of weight 2 by Lemma 6.2 and three words of weight 3. This yields

$$|C| \leq 1 + 1 + n - 1 + 3 = n + 4 \leq 2n \text{ for } n \geq 4.$$

Suppose there is a word $y \in C$ of weight 3, $y \notin \{u, v, z\}$. Then y is uniquely determined,

$$y = (0,1,1,1,0,...,0),$$

and u, v, z, y are all words of weight 3 in C. Observing that in this situation C has no word of weight 1 the above estimate modifies to

$$|C| \leq 1 + 0 + n - 1 + 4 = n + 4 \leq 2n \text{ for } n \geq 4.$$

□

Theorem 6.3. *Let $m, n, k \in \mathbb{N}$, $m \geq 2$, $n > k \geq 1$. Then we have*

$$\omega(\overline{HG}(m, n, n-k)) = m$$

for every $n \geq \frac{1}{2}m(m+1)(k-1) + 1$.

Proof. As $H = \{(a, ..., a) : 0 \leq a \leq m-1\}$ induces a clique in $\overline{HG}(m, n, n-k)$, we have

$$\omega(\overline{HG}(m, n, n-k)) \geq |H| = m.$$

Let C be a maximal clique in $\overline{HG}(m, n, n-k)$, $|C| \geq m$. By definition any two different words in C have the same entries in at most $k-1$ positions. By Proposition 1.1, we may assume $\overline{0} \in C$. Let $w_0 = \overline{0}$. Word w_0 has all entries 0 from position $p_0 = 1$ on.
Let $w_1 \in C$, $w_1 \neq w_0$. As a neighbor of w_0 word w_1 has at most $k-1$ entries 0. We may assume w.l.o.g. that all entries of w_1 from position $p_1 = p_0 + (k-1)$ on are 1.
Let $w_2 \in C$, $w_2 \notin \{w_0, w_1\}$. From position p_1 on we have in w_2 at most $(k-1)$ entries 0 and at most $k-1$ entries 1. We may assume that all entries of w_2 from position $p_2 = p_1 + 2(k-1)$ on are 2.
We may go on this way until we arrive at word w_{m-1} and position p_{m-1}. Now the words $w_j \in C$, $0 \leq j \leq m-1$, have the following property from position p_{m-1} on: w_j has only entries j.
Suppose w_m is another word in C, $w_m \notin \{w_0, ..., w_{m-1}\}$. Then w_m has from position p_{m-1} on at most $(k-1)$ entries 0, at most $(k-1)$ entries 1,..., at most $(k-1)$ entries $m-1$. This implies

$$n < n_0 = p_m = p_{m-1} + m(k-1).$$

If $n \geq n_0$, then there is no space for $w_m \in C$, i.e

$$C = \{w_0, ..., w_{m-1}\}, \ |C| = m.$$

We have

$$p_j - p_{j-1} = j(k-1) \text{ for } 1 \leq j \leq m, \ p_0 = 1,$$

which implies

$$\begin{aligned} p_m - p_0 &= \sum_{j=1}^{m}(p_j - p_{j-1}) \\ &= (k-1)\sum_{j=1}^{m} j = \frac{1}{2}m(m+1)(k-1), \end{aligned}$$

and

$$n_0 = p_m = \frac{1}{2}m(m+1)(k-1) + 1.$$

□

The following result was partially proved by Jamison and Matthews [7] for $k = 1$ and $k = 2$.

Theorem 6.4. *Let $m, n, k \in \mathbb{N}$, $m \geq 2$, $n > k \geq 1$. For $n \geq \frac{1}{2}m(m+1)(k-1)+1$ holds*

$$\chi(HG(m, n, n-k)) = m^{n-1}.$$

Proof. For $n \geq \frac{1}{2}m(m+1)(k-1)+1$, we have by Theorem 6.3

$$\alpha(HG(m, n, n-k)) = m$$

which implies

$$\chi(HG(m, n, n-k)) \geq \frac{|V(HG(m, n, n-k))|}{\alpha(HG(m, n, n-k))} = \frac{m^n}{m} = m^{n-1}.$$

We show that

$$\chi(HG(m, n, n-k)) \leq m^{n-1}.$$

Define the labeling f by coloring vertex $x = (x_1, ..., x_n)$ with color

$$f(x) = (x_2 + x_1, x_3 + x_1, ..., x_n + x_1) \pmod{m}.$$

Suppose that $x \neq y$ are adjacent vertices of $HG(m, n, n-k)$, $d(x, y) \leq n-k \leq n-1$. Then there is at least one position j $(1 \leq j \leq n)$ such that $x_j = y_j$.
Let $f(x) = f(y)$. Then we have

$$(x_2 + x_1, x_3 + x_1, ..., x_n + x_1) = (y_2 + y_1, y_3 + y_1, ..., y_n + y_1) \pmod{m}$$

and we conclude

$$x_i + x_1 = y_i + y_1 (\bmod m) \text{ for } 2 \leq i \leq n.$$

Assume $x_1 \neq y_1$. Then there is some j, $2 \leq j \leq n$, with $x_j = y_j$. Now $x_j + x_1 = y_j + y_1 \bmod m$ implies $x_1 = y_1$, which is a contradiction.
So we have $x_1 = y_1$ and $x_i + x_1 = y_i + y_1 (\bmod m)$ for every $i = 2, ..., n$ implies $x = y$, which is again a contradiction.
Therefore f is a proper coloring of the vertices of $HG(m, n, n-k)$ with at most m^{n-1} colors, which implies

$$\chi(HG(m, n, n-k)) \leq m^{n-1}.$$

□

Lemma 6.3. *Let $m, n, d, m \geq 2, n \geq d$, be positive integers. Then we have*

$$\chi(\overline{HG}(m, n, d)) \leq m^{n-d}.$$

Proof. We define the labeling f of $x = (x_1, ..., x_n)$ by $f(x) = (x_1, ..., x_{n-d})$. Suppose that x and y are adjacent vertices in $\overline{HG}(m, n, d)$, $d(x, y) > d$. We show that $f(x) \neq f(y)$.
Suppose $f(x) = f(y)$. Then we have

$$(x_1, ..., x_{n-d}) = (y_1, ..., y_{n-d}),\ x_1 = y_1, ..., x_{n-d} = y_{n-d},$$

and $d(x, y) \leq d$, which is a contradiction. The number of colors in this proper coloring is equal to m^{n-d}. Therefore we have

$$\chi(\overline{HG}(m, n, d)) \leq m^{n-d}.$$

□

Theorem 6.5. *Let m, n, d be positive integers, $m \geq 2$ and $d = n$ or $d = n - 1 \geq 1$. Then we have*

1. $\chi(HG(m, n, d)) = \omega(HG(m, n, d)) = m^d$,

2. $\chi(\overline{HG}(m, n, d)) = \omega(\overline{HG}(m, n, d)) = m^{n-d}$.

Proof.

1. If $d = n$, then $HG(m, n, d)$ is the complete graph on m^n vertices and the statements are obviouly true. So we assume $d = n - 1 \geq 1$.
By Lemma 6.1 we have

$$\omega(HG(m, n, n - 1)) \geq m^{n-1}.$$

Also Theorem 6.4 implies

$$\chi(HG(m, n, n - 1)) = m^{n-1}.$$

Therefore we have

$$m^{n-1} \leq \omega(HG(m, n, n - 1)) \leq \chi(HG(m, n, n - 1)) = m^{n-1}$$

and

$$\chi(HG(m, n, n - 1)) = \omega(HG(m, n, n - 1)) = m^{n-1}.$$

2. We have by Lemma 6.3 and Theorem 6.3

$$m = \omega(\overline{HG}(m, n, n - 1)) \leq \chi(\overline{HG}(m, n, n - 1)) \leq m,$$

which implies

$$\chi(\overline{HG}(m, n, n - 1)) = \omega(\overline{HG}(m, n, n - 1)) = m.$$

□

Corollary 6.1. *For every $m \in \mathbb{N}$, $m \geq 2$, we have*

1. $\chi(HG(m,3,2)) = \omega(HG(m,3,2)) = m^2$,

2. $\chi(\overline{HG}(m,3,2)) = \omega(\overline{HG}(m,3,2)) = m$.

Theorem 6.6. *For every $m \geq 2$ we have*

$$\chi(\overline{HG}(m,4,2)) = m^2.$$

Proof. For $m = 2$ we have by Lemma 6.3

$$\chi(\overline{HG}(2,4,2)) \leq 4.$$

We conclude by Theorem 6.1(1)

$$\alpha(\overline{HG}(2,4,2)) = \omega(HG(2,4,2)) = 5.$$

which implies

$$3.2 = \frac{2^4}{5} = \frac{|V(\overline{HG}(2,4,2))|}{\alpha(\overline{HG}(2,4,2))} \leq \chi(\overline{HG}(2,4,2)) \leq 4$$

and

$$\chi(\overline{HG}(2,4,2)) = 4.$$

For $m \geq 3$ we have by Theorem 6.1(2)

$$\alpha(\overline{HG}(m,4,2)) = \omega(HG(m,4,2)) = m^2$$

and we conclude by Lemma 6.3

$$m^2 = \frac{m^4}{m^2} = \frac{|V(\overline{HG}(m,4,2))|}{\alpha(\overline{HG}(m,4,2))} \leq \chi(\overline{HG}(m,4,2)) \leq m^2,$$

$$\chi(\overline{HG}(m,4,2)) = m^2.$$

□

Now we use some ideas from the theory of error-correcting codes to provide bounds on the chromatic number of Hamming graphs. We begin by recalling the necessary terminology, see [4].

We assume now that the alphabet is the Galois field $GF(q) = \mathbb{F}_q$, where q is a prime power, and we regard ${\mathbb{F}_q}^n$ as an n-dimensional vector space over $\mathbb{F}_q$. If C is a k-dimensional subspace of ${\mathbb{F}_q}^n$, then C is called a linear $[n,k]$-*code*, or sometimes, if we wish to specify also the minimum distance d of C, an $[n, k, d]$-*code*. The *minimum distance*, denoted $d(C)$, is defined to be the smallest of the distances between distinct codewords. That is,

$$d(C) = min\{d(x, y) : x, y \in C, x \neq y\}.$$

Suppose that C is an $[n, k]$-*code* over $GF(q)$ and a is any vector in ${\mathbb{F}_q}^n$. Then the set $a + C$ defined by

$$a + C = \{a + x : x \in C\}$$

is the *coset* of C represented by a. Every coset of C contains exactly $|C| = q^k$ codewords and any two cosets are disjoint.
The *dual code* of C, denoted by $C^{\perp}$, is defined to be the set of those vectors of ${\mathbb{F}_q}^n$ which are orthogonal to every codeword of C, i.e.

$$C^{\perp} = \{v \in {\mathbb{F}_q}^n : v \cdot u = 0 \text{ for all } u \in C\},$$

where $v \cdot u$ denotes the inner product $v(1)u(1) + ... + v(n)u(n)$. We have $(C^{\perp})^{\perp} = C$. The following proposition states that $C^{\perp}$ is a linear code of dimension $n - k$.

Proposition 6.1. [4] *Suppose C is an $[n, k]$-code over $GF(q)$. Then the dual code $C^{\perp}$ of C is a linear $[n, n-k]$-code.*

A $k \times n$ matrix whose rows form a basis of a linear $[n, k]$-code is called a *generator matrix* of the code. A *parity-check matrix H* for an $[n, k]$-code C is a generator matrix of $C^{\perp}$. If H is a parity-check matrix of C then

$$C = \{x \in {\mathbb{F}_q}^n : xH^T = \overline{0}\}$$

where H^T denotes the transpose of H and $\overline{0}$ is the zero word.
The following proposition establishes a fundamental relationship between the minimum distance of a linear code and a linear independence property of the columns of a parity-check matrix.

Proposition 6.2. [4] *Suppose C is a linear $[n, k]$-code over $GF(q)$ with parity-check matrix H. Then the minimum distance of C is d if and only if any $d - 1$ columns of H are linearly independent but some d columns are linearly dependent.*

The following theorem extends part 2 of Theorem 6.1.

Theorem 6.7. *If $2 \leq n \leq m + 1$ and m is prime power we have*
1. $\chi(HG(m, n, 2)) = \omega(HG(m, n, 2)) = m^2$,

2. $\chi(\overline{HG}(m, n, 2)) = \omega(\overline{HG}(m, n, 2)) = m^{n-2}$.

Proof.

1. Let $\mathbb{F}_m = \{a_0 = 0, a_1 = 1, a_2, ..., a_{m-1}\}$ be the finite field with m elements. For $n = 2$ the Hamming graph $HG(m, n, 2)$ is the complete graph on m^2 vertices. As the assertions become obvious in this case, we assume $2 < n \le m + 1$.
Define the matrix H by

$$H = \begin{pmatrix} 0 & 1 & 1 & . & . & . & 1 \\ 1 & 0 & a_1 & . & . & . & a_{n-2} \end{pmatrix}.$$

Clearly, H has rank 2. So H is a parity-check matrix for a $[n, n-2]$-code C. The following table shows all types of column pairs of H and their determinants, $1 \le i, j \le n - 2$.

Columns	$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\a_i\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\a_i\end{pmatrix}$	$\begin{pmatrix}1\\a_i\end{pmatrix}, \begin{pmatrix}1\\a_j\end{pmatrix}$
det (A)	-1	-1	a_i	$a_j - a_i$

As *det* $(A) \ne 0$, any two columns of H are linearly independent. The first three columns of H are linearly dependent.
By Proposition 6.2 we conclude that the $[n, n-2]$-code C has minimum distance 3. We may take the cosets of C as color classes for $HG(m, n, 2)$. As $|C| = m^{n-2}$, we get

$$\chi(HG(m, n, 2)) \le \frac{m^n}{m^{n-2}} = m^2.$$

By Theorem 6.1(2) we conclude

$$m^2 = \omega(HG(m, n, 2)) \le \chi(HG(m, n, 2)) \le m^2,$$

$$\chi(HG(m, n, 2)) = \omega(HG(m, n, 2)) = m^2.$$

2. Code C in part 1 induces a clique in $\overline{HG}(m, n, 2)$, thus

$$\omega(\overline{HG}(m, n, 2)) \ge |C| = m^{n-2}.$$

But by Lemma 6.3 we have

$$m^{n-2} \le \omega(\overline{HG}(m, n, 2)) \le \chi(\overline{HG}(m, n, 2)) \le m^{n-2},$$

$$\chi(\overline{HG}(m, n, 2)) = \omega(\overline{HG}(m, n, 2)) = m^{n-2}.$$

□

The following theorem shows that in Theorem 6.7 the size m of the alphabet needs not always be a prime power.

Theorem 6.8. *Let $p_1 < p_2 < \ldots$ be the sequence of all primes. Suppose $m \geq 2$ is not divisible by $p_1, \ldots, p_k$. Then for every n, $2 \leq n \leq p_{k+1} + 1$, we have*

1. $\chi(HG(m, n, 2)) = \omega(HG(m, n, 2)) = m^2$,

2. $\chi(\overline{HG}(m, n, 2)) = \omega(\overline{HG}(m, n, 2)) = m^{n-2}$.

Proof. As the statements of the theorem are trivially true for $n = 2$, we assume $n > 2$.
In this proof we return to the alphabet $\mathbb{Z}_m$. As $\mathbb{Z}_m$, considered as the ring of integers modulo m, needs not be a field we are more cautious with arguments from coding theory.

1. The numbers $1, 2, \ldots, n-2$ are consecutive prime residues modulo m. Define the matrix $H \in {\mathbb{Z}_m}^{2\times n}$ by

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & . & . & . & 1 \\ 1 & 0 & 1 & 2 & . & . & . & n-2 \end{pmatrix}.$$

We take H as the parity-check matrix of a code $C \subseteq {\mathbb{Z}_m}^n$, i.e.

$$C = \{x \in {\mathbb{Z}_m}^n : xH^T = \overline{0}\}.$$

The words $x = (x_1, \ldots, x_n)$ of C satisfy in $\mathbb{Z}_m$ the equations

$$\begin{aligned} x_2 + x_3 + x_4 + \ldots + x_n &= 0, \\ x_1 + x_3 + 2x_4 + \ldots + (n-2)x_n &= 0. \end{aligned}$$

Clearly, x_1 and x_2 are uniquely determined for every choice of $x_3, \ldots, x_n$. Therefore, we have $|C| = m^{n-2}$. The code C contains the zero word $\overline{0}$, but no word of weight 1.
Moreover, C contains no word $(x_1, \ldots, x_n)$ of weight 2 with $x_1 \neq 0$ or $x_2 \neq 0$. Suppose C contains a word $x = (x_1, \ldots, x_n)$ of weight 2, e.g. $x_j \neq 0$, $x_k \neq 0$, $2 < j < k \leq n$, $x_i = 0$ for every $i \neq j, k$. Then we have in $\mathbb{Z}_m$

$$x_j + x_k = 0, \;\; (j-2)x_j + (k-2)x_k = 0.$$

Inserting $x_k = -x_j$ in the second equation yields

$$(j-k)x_j = 0.$$

As $j-k$ is coprime to m, it has a multiplicative inverse in $\mathbb{Z}_m$, which implies $x_j = 0$ in contradiction to our asumption.
The word $(1, 1, -1, 0, ..., 0)$ is in C and has weight 3. So the minimal distance $d(x, y) = w(x-y)$ of words $x \neq y$ in C is 3.
Taking the cosets of C as color classes for $HG(m, n, 2)$, we see

$$\chi(HG(m, n, 2)) \leq \frac{m^n}{m^{n-2}} = m^2.$$

But $n \leq m+1$ and by Theorem 6.1(2) we conclude

$$m^2 = \omega(HG(m, n, 2)) \leq \chi(HG(m, n, 2)) \leq m^2,$$

$$\chi(HG(m, n, 2)) = \omega(HG(m, n, 2)) = m^2.$$

2. Code C in part 1 induces a clique in $\overline{HG}(m, n, 2)$, thus

$$\omega(\overline{HG}(m, n, 2)) \geq |C| = m^{n-2}.$$

But by Lemma 6.3 we have

$$m^{n-2} \leq \omega(\overline{HG}(m, n, 2)) \leq \chi(\overline{HG}(m, n, 2)) \leq m^{n-2},$$

$$\chi(\overline{HG}(m, n, 2)) = \omega(\overline{HG}(m, n, 2)) = m^{n-2}.$$

□

The following corollary supplements the result in Theorem 6.6.

Corollary 6.2. *For every odd m we have*
1. $\chi(HG(m, 4, 2)) = \omega(HG(m, 4, 2)) = m^2$,

2. $\chi(\overline{HG}(m, 4, 2)) = \omega(\overline{HG}(m, 4, 2)) = m^2$.

Proof. The two parts follow from Theorem 6.8 with $p_1 = p_k = 2$. □

The *Hamming code Ham*(r, m), $r \in \mathbb{N}$, m a prime power, is a linear code over the finite field $\mathbb{F}_m$ (see [4]). It has word length

$$n = \frac{m^r - 1}{m - 1},$$

dimension $n - r$ and minimal distance 3.
The following result, except for the clique number, was obtained by Jamison and Matthews [7].

Theorem 6.9. *If m is a prime power and $r \in \mathbb{N}$, $r \geq 2$, then*

$$\chi(HG(m, \frac{m^r - 1}{m - 1}, 2)) = \omega(HG(m, \frac{m^r - 1}{m - 1}, 2)) = m^r.$$

Proof. Let $\mathbb{F}_m$ be the alphabet and C the Hamming code $Ham(r, m)$, C is an $[n, n-r, 3]$-code. We have

$$n = \frac{m^r - 1}{m - 1}, \ |C| = m^{n-r}, \ d(C) = 3.$$

Take the cosets of C as color classes, therefore

$$\chi(HG(m, \frac{m^r - 1}{m - 1}, 2)) \leq \frac{m^n}{m^{n-r}} = m^r.$$

By Theorem 6.1(1) we conclude

$$m^r = n(m-1) + 1 = \omega(HG(m, \frac{m^r - 1}{m - 1}, 2)) \leq \chi(HG(m, \frac{m^r - 1}{m - 1}, 2)) \leq m^r,$$

$$\chi(HG(m, \frac{m^r - 1}{m - 1}, 2)) = \omega(HG(m, \frac{m^r - 1}{m - 1}, 2)) = m^r.$$

□

Corollary 6.3. *If m is a prime power then*

$$\chi(HG(m, m+1, 2)) = \omega(HG(m, m+1, 2)) = m^2.$$

Proof. This follows by Theorem 6.9 with $r = 2$. □

Corollary 6.4. *For every $r \in \mathbb{N}$, $r \geq 2$, we have*

$$\chi(HG(2, 2^r - 1, 2)) = \omega(HG(2, 2^r - 1, 2)) = 2^r.$$

Proof. This follows by Theorem 6.9 with $m = 2$. □

The authors [6] showed that $\chi(HG(2, 2^r - 2, 2)) = 2^r$. But in this case we have $\omega(HG(2, 2^r - 2, 2)) = 2^r - 1$ for $r \geq 3$ and the clique number is less than the chromatic number.

Denote by $max_s(r, q)$, $1 \leq s \leq r$, the maximal number of vectors in ${\mathbb{F}_q}^r$ such that any s of them are linearly independent. There are some values of $max_s(r, q)$ known, see [4].

Proposition 6.3.

1. *$max_3(r, 2) = 2^{r-1}$ for $r \geq 3$,*

2. $max_3(3, q) = \begin{cases} q+1 & \textit{if } q \textit{ is odd} \\ q+2 & \textit{if } q \textit{ is even} \end{cases}.$

Theorem 6.10. *Let q be a prime power and $n = max_s(r, q)$. Then we have*

$$\chi(HG(q, n, s)) \leq q^r.$$

Proof. Take the $n = max_s(r, q)$ vectors of $\mathbb{F}_q{}^r$ as the columns of a matrix H. Then H is a parity-check matrix for a $[n, n-r]$-code C. By Proposition 6.2 we conclude that the $[n, n-r]$-code C has minimum distance $d(C) \geq s+1$. We may take the cosets of C as color classes of $HG(q, n, s)$. As $|C| = q^{n-r}$, we get

$$\chi(HG(q, n, s)) \leq \frac{q^n}{q^{n-r}} = q^r.$$

□

The following result, except for the clique number, has been shown by Kim, Du and Pardalos [9].

Theorem 6.11. *For every $r \in \mathbb{N}$, $r \geq 3$, we have*

$$\chi(HG(2, 2^{r-1}, 3)) = \omega(HG(2, 2^{r-1}, 3)) = 2^r.$$

Proof. By Proposition 6.3(1) and Theorem 6.10 we have

$$\chi(HG(2, 2^{r-1}, 3)) \leq 2^r.$$

Theorem 6.2 implies

$$2^r = \omega(HG(2, 2^{r-1}, 3)) \leq \chi(HG(2, 2^{r-1}, 3)) \leq 2^r,$$

$$\chi(HG(2, 2^{r-1}, 3)) = \omega(HG(2, 2^{r-1}, 3)) = 2^r.$$

□

Theorem 6.12. *Let q be a prime power and* $\varepsilon = \begin{cases} 0 & \textit{if } q \textit{ is odd} \\ 1 & \textit{if } q \textit{ is even} \end{cases}$. *Then we have*

$$\chi(HG(q, q+1+\varepsilon, 3)) = \omega(HG(q, q+1+\varepsilon, 3)) = q^3.$$

Proof. By Proposition 6.3(2) and Theorem 6.10 we have

$$\chi(HG(q, q+1+\varepsilon, 3)) \leq q^3.$$

Lemma 6.1 implies

$$q^3 \leq \omega(HG(q, q+1+\varepsilon, 3)) \leq \chi(HG(q, q+1+\varepsilon, 3)) \leq q^3,$$

$$\chi(HG(q, q+1+\varepsilon, 3)) = \omega(HG(q, q+1+\varepsilon, 3)) = q^3.$$

□

Chapter 7

Conclusion

In the previous chapters a number of interesting results have been obtained. However, some questions remain unanswered and may possibly be the starting point for future research. We showed that the ω_0-conjecture is true for distance parameter $d \leq 6$ and in the binary case, but in general it is an open problem for $d \geq 7$. To complete the proof of the ω_0-conjecture one could try to extend the techniques used for $d \leq 6$. In this thesis we spent no effort on evaluating or estimating the constant $n_0 = n_0(m, d)$ in Theorem 1.1. So we may ask for a good upper bound of $n_0(m, d)$.

As only few exact values of the chromatic number $\chi(HG(m, n, d))$ are known, many open question remain. Wan [12] conjectured

$$\chi(HG(2, n, 2)) = 2^{\lceil \log_2(n+1) \rceil}.$$

But this has been proved only, if $n+1$ or $n+2$ is a power of 2 [6]. It might be true that $\chi(HG(m, n, d)) = \omega_0(m, n, d)$ for infinitely many n. For $d = 2$ this follows from Theorem 6.9 (chapter 6). For $d = 3$ and $m = 2$ it is confirmed by a result of Kim et al. [9]. Another coloring problem for ${K_m}^n$ arises, if the Hamming distance is replaced by the euclidean distance, see Fertin et al. [2].

An important task of coding theory is to construct 'good' codes with many words and guaranteed minimal distance. A good code in this sense, which can be drawn from $HG(m, n, d)$, is represented by a maximal set of independent vertices. Therefore, it would be interesting to know more about the independence number $\alpha(HG(m, n, d))$.

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